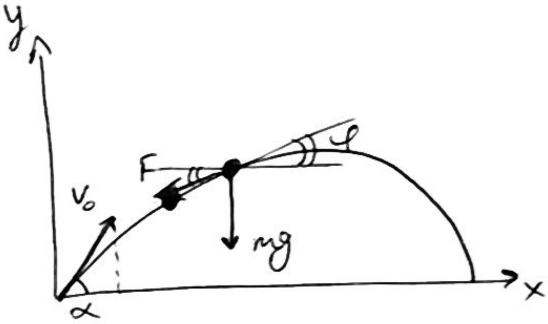
Задача. Тело бросили под углом α с начальной скоростью V_0. $F_{сопр} = kV$ (k дано). Найти закон движения тела. Масса m.

Дано:
V_0, α, g k, m
$x(t)-?$ $y(t)-?$

$$\begin{cases} -m\ddot{x} = F\cdot\cos\varphi & (1) \\ -m\ddot{y} = mg + F\sin\varphi & (2) \\ F = k\cdot\sqrt{\dot{x}^2+\dot{y}^2} & (3) \\ tg\varphi = \dfrac{\dot{y}}{\dot{x}} & (4) \end{cases}$$



1) Выразим $\cos\varphi$ и $\sin\varphi$ через $tg\varphi$.

$$1+tg^2\varphi = \frac{\sin^2\varphi+\cos^2\varphi}{\cos^2\varphi} = \frac{1}{\cos^2\varphi} \Rightarrow \cos\varphi = \sqrt{\frac{1}{1+tg^2\varphi}}$$

$$\sin\varphi = \sqrt{1-\cos^2\varphi} = \sqrt{1-\frac{1}{1+tg^2\varphi}} = \sqrt{\frac{1+tg^2\varphi-1}{1+tg^2\varphi}} = \frac{tg\varphi}{\sqrt{1+tg^2\varphi}}$$

Подставим в (1, 2).

$$-m\ddot{x} = k\sqrt{\dot{x}^2+\dot{y}^2}\cdot\sqrt{\frac{1}{1+tg^2\varphi}} = k\sqrt{\dot{x}^2+\dot{y}^2}\cdot\sqrt{\frac{1}{1+\frac{\dot{y}^2}{\dot{x}^2}}} =$$

$$= k\sqrt{\frac{\dot{x}^2+\dot{y}^2}{\frac{\dot{x}^2+\dot{y}^2}{\dot{x}^2}}} = k\dot{x}$$

$$-m\ddot{y} = mg + k\sqrt{\dot{x}^2+\dot{y}^2}\cdot\frac{\dot{y}/\dot{x}}{\sqrt{1+\dot{y}^2/\dot{x}^2}} = mg + k\sqrt{\frac{\dot{x}^2+\dot{y}^2}{\frac{\dot{x}+\dot{y}^2}{\dot{x}^2}}}\cdot\frac{\dot{y}}{\dot{x}} =$$

$$= mg + k\dot{y}$$

$$\begin{cases} -\ddot{x} = \dfrac{k}{m}\dot{x} & \text{Д.У. 1} \\ -\ddot{y} = g + \dfrac{k}{m}\dot{y} & \text{Д.У. 2} \end{cases}$$

решив эти диф. ур., найдём законы движения $x(t)$ и $y(t)$.

2) Решаем ДУ 1

$$-\ddot{x} = \frac{k}{m}\dot{x}$$

$$-\frac{d^2x}{dt^2} = \frac{k}{m}\frac{dx}{dt}$$

$$\int \frac{d^2x}{dt^2} = -\frac{k}{m}\int \frac{dx}{dt}$$

$$\frac{dx}{dt} = -\frac{k}{m}x + C_1$$

$$\frac{dx}{-\frac{k}{m}x + C_1} = dt$$

$$\int \frac{d\left(-\frac{k}{m}x + C_1\right)}{-\frac{k}{m}x + C_1}\left(-\frac{m}{k}\right) = \int dt$$

$$-\frac{m}{k}\cdot \ln\left(-\frac{k}{m}x + C_1\right) = t + C_2$$

$$\ln\left(-\frac{k}{m}x + C_1\right) = \left(-\frac{k}{m}\right)t + \left(-\frac{k}{m}\right)C_2$$

$$-\frac{k}{m}x + C_1 = e^{-\left(\frac{k}{m}t + \frac{k}{m}C_2\right)} = e^{\frac{k}{m}t}\cdot \underbrace{e^{\frac{k}{m}C_2}}_{C} = C\cdot e^{\frac{k}{m}t}$$

$$-\frac{k}{m}x + C_1 = Ce^{-\frac{k}{m}t}$$

$$-\frac{k}{m}x = Ce^{-\frac{k}{m}t} - C_1$$

$$-x = \underbrace{\frac{m}{k}C}\, e^{-\frac{k}{m}t} - \underbrace{\frac{m}{k}C_1}$$

$$-x = C\cdot e^{-\frac{k}{m}t} - C_1$$

Нач. условие: $x(0) = 0$.

$0 = C - C_1 \Rightarrow C_1 = C$

$$x(t) = -C\left(e^{-\frac{k}{m}t} - 1\right) \quad (2.1)$$

$$\left.\begin{aligned}\dot{x} &= C\cdot e^{-\frac{k}{m}t}\left(-\frac{k}{m}\right)\\ \dot{x}(0) &= V_0\cdot\cos\alpha\end{aligned}\right\} \Rightarrow V_0\cos\alpha = C\left(-\frac{k}{m}\right) \Rightarrow C = -V_0\cos\alpha\cdot\frac{m}{k}$$

$\rightarrow (2.1)$

$$\boxed{x(t) = V_0\cdot\cos\alpha\cdot\frac{m}{k}\left(e^{-\frac{k}{m}t} - 1\right)}$$

закон движ. $x(t)$ найден.

3) Решим Д.У. 2

$$-\ddot{y} = g + \frac{k}{m}\dot{y}$$

$$-\int \frac{d^2y}{dt} = \int\left(g + \frac{k}{m}\frac{dy}{dt}\right)$$

$$-\frac{dy}{dt} = gt + \frac{k}{m}y + C$$

линейное неоднородное диф. ур. первого порядка. Решаем методом вариации произвольной константы.

(3.1) $-\dot{y} - \frac{k}{m}y = gt + C$

3.a.). Решим $-\dot{y} - \frac{k}{m}y = 0$

$$-\frac{dy}{dt} = \frac{k}{m}y$$

$$-\int\frac{dy}{y} = \frac{k}{m}\int dt$$

$$\ln y = -\frac{k}{m}t + \tilde{C}$$

$$y = e^{-(\frac{k}{m}t + \tilde{C})} = e^{-\frac{k}{m}t}\cdot e^{\tilde{C}} = \tilde{C}\cdot e^{-\frac{k}{m}t}$$

] $\tilde{C} = f(t)$, тогда

(3.2)

подст. в (3.1)

$$\begin{cases} y = f\cdot e^{-\frac{k}{m}t}, \\ y' = f'\cdot e^{-\frac{k}{m}t} + f\cdot\left(e^{-\frac{k}{m}t}\right)' = f'\cdot e^{-\frac{k}{m}t} + f\left(-\frac{k}{m}\right)e^{-\frac{k}{m}t} \end{cases}$$

$$-f' e^{-\frac{k}{m}t} + \cancel{f\cdot\frac{k}{m}\cdot e^{-\frac{k}{m}t}} - \cancel{\frac{k}{m}f e^{-\frac{k}{m}t}} = gt + C$$

$$f'\cdot e^{-\frac{k}{m}t} = -gt + C$$

$$\frac{df}{dt} = -e^{\frac{k}{m}t}(gt + C)$$

$$\int df = -\int e^{\frac{k}{m}t}(gt + C)\,dt$$

(3.3) $$f = -\underbrace{g\int e^{\frac{k}{m}t}\cdot t\,dt}_{I_1} - \underbrace{C\int e^{\frac{k}{m}t}\,dt}_{I_2}.$$

3.б). Считаем I_1.

$$I_1 = -g\int e^{\frac{k}{m}t}\cdot t\,dt = -g\frac{m}{k}\int t\cdot d\left(e^{\frac{k}{m}t}\right) =$$

$$= -g\frac{m}{k}\cdot\left[t\cdot e^{\frac{k}{m}t} - \int e^{\frac{k}{m}t}dt\right] =$$

$$= -g\frac{m}{k}\left[t\cdot e^{\frac{k}{m}t} - \int e^{\frac{k}{m}t}\cdot\left(\frac{m}{k}\right)d\left(\frac{k}{m}t\right)\right] =$$

$$= -g\frac{m}{k}\left[t\cdot e^{\frac{k}{m}t} - \frac{m}{k}e^{\frac{k}{m}t} + C_1\right]$$

3.в) Считаем I_2.

$$I_2 = C\int e^{\frac{k}{m}t}dt = C\left(\frac{m}{k}e^{\frac{k}{m}t} + C_2\right) =$$

$$= C\frac{m}{k}e^{\frac{k}{m}t} + C_2$$

3.г) Находим f из (3.3):

$$f = I_1 + I_2 = -g\frac{m}{k}te^{\frac{k}{m}t} + g\frac{m^2}{k^2}e^{\frac{k}{m}t} - C_1 g\frac{m}{k} + C\frac{m}{k}e^{\frac{k}{m}t} + C_2$$

f подст. в (3.2):

$$y = f\cdot e^{-\frac{k}{m}t} = -g\frac{m}{k}t + \overbrace{g\frac{m^2}{k^2}}^{const} - C_1 g\frac{m}{k}e^{-\frac{k}{m}t} + \overbrace{C\frac{m}{k}}^{const} + C_2 e^{-\frac{k}{m}t} =$$

$$= -g\frac{m}{k}t + e^{-\frac{k}{m}t}\left(C_2 + C_1 g\frac{m}{k}\right) + C =$$

$$= C_1\cdot e^{-\frac{k}{m}t} - g\frac{m}{k}t + C$$

Нач. условия: $y(0) = 0 \Rightarrow C_1 = -C$

$$y(t) = C_1\left(e^{-\frac{k}{m}t} - 1\right) - \frac{gm}{k}t$$

$$\dot{y} = C_1\frac{-k}{m}e^{-\frac{k}{m}t} - \frac{gm}{k}\ ;\quad \dot{y}(0) = V_0\sin\alpha$$

$$-C_1\frac{k}{m} - \frac{m}{k}g = V_0\sin\alpha \Rightarrow C_1 = -\left(V_0\sin\alpha + \frac{m}{k}g\right)\frac{m}{k}$$

$$\boxed{y(t) = -\left(V_0\sin\alpha + \frac{m}{k}g\right)\frac{m}{k}\left(e^{-\frac{k}{m}t} - 1\right) - \frac{gm}{k}t}$$

4). $$\begin{cases} x(t) = -V_0 \cos\alpha \dfrac{m}{k}\left(e^{-\frac{k}{m}t} - 1\right) \\ y(t) = \left(V_0 \sin\alpha + \dfrac{m}{k}g\right)\dfrac{m}{k}\left(e^{-\frac{k}{m}t} - 1\right) - \dfrac{gm}{k}t \end{cases}$$

5) Найдём время полёта τ : $y=0$

$$-\left(V_0 \sin\alpha + \frac{mg}{k}\right)\left(e^{-\frac{k}{m}\tau} - 1\right) = g\tau$$