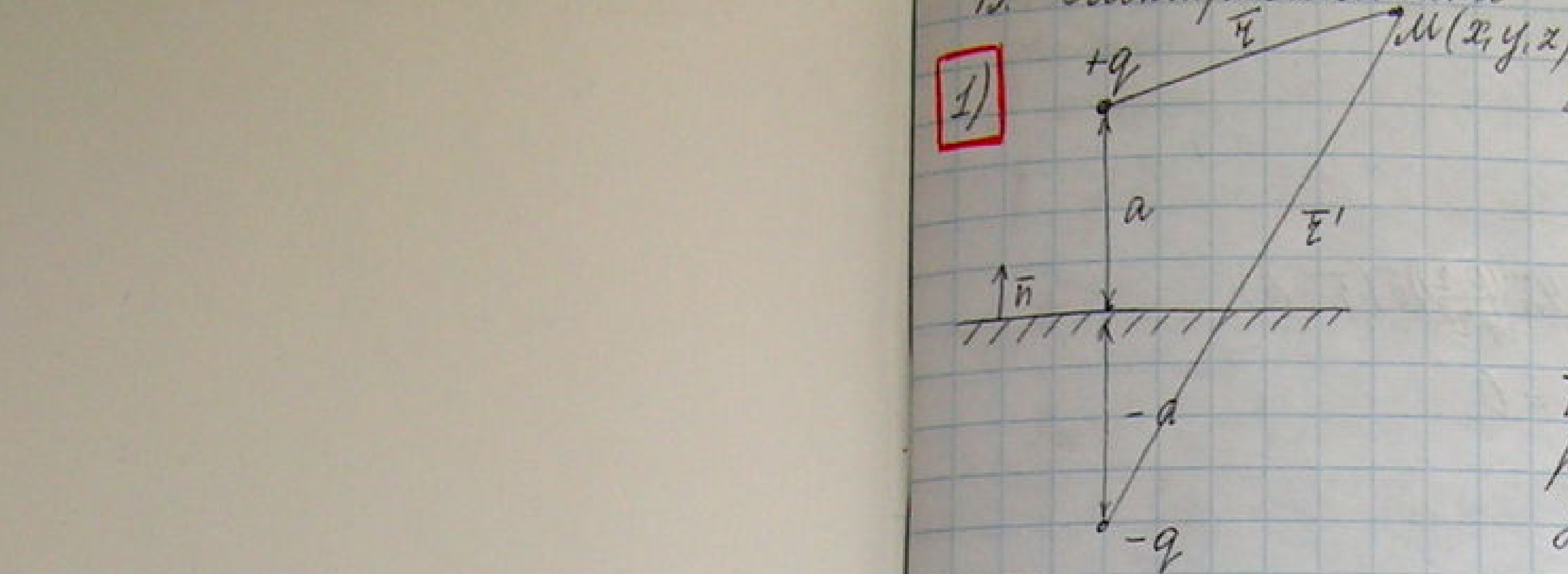
15. Электростатика проводников.

1)



$$\Delta\varphi = -4\pi q\delta(\bar{r}-\bar{a}), \quad (z>0)$$

$$\varphi\Big|_{\Sigma} = 0 \qquad (\Sigma:\ z=0;\ x,y\ \forall)$$

$\varphi \Rightarrow 0$ на бесконечн.

Т.к. функ Грина $G(M,M_0)$ – решение следующей краевой задачи: $\Delta_M G = -\delta(M,M_0)$,

$$G\Big|_{M\in\Sigma} = 0, \text{ то}$$

$$\varphi = 4\pi q G = \frac{q}{\sqrt{x^2+y^2+(z-a)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+a)^2}} =$$

$$= \frac{q}{r} - \frac{q}{r'}.$$

Плотность пов. зарядов:

$$\sigma_s = -\frac{1}{4\pi}\frac{\partial\varphi}{\partial n}\Big|_{z=0} = -\frac{1}{4\pi}\left\{\frac{-q(z-a)}{(\rho^2+(z-a)^2)^{3/2}} - \frac{-q(z+a)}{(\rho^2+(z+a)^2)^{3/2}}\right\}_{z=0}$$

$$= -\frac{qa}{2\pi(\rho^2+a^2)^{3/2}}.$$

Полный индуциров. заряд на плоскости:

$$\oint \sigma_s\, ds = -\frac{1}{4\pi}\oint \frac{\partial\varphi}{\partial n}\Big|_{z=0} ds = -\frac{qa}{2\pi}\int_0^{2\pi} d\varphi \int_0^{\infty}\frac{\rho\, d\rho}{(\rho^2+a^2)^{3/2}} =$$

$$= -\frac{qa}{2}\int_0^{\infty}\frac{d(\rho^2+a^2)}{(\rho^2+a^2)^{3/2}} = \frac{qa}{(\rho^2+a^2)^{1/2}}\Big|_0^{\infty} = -q.$$

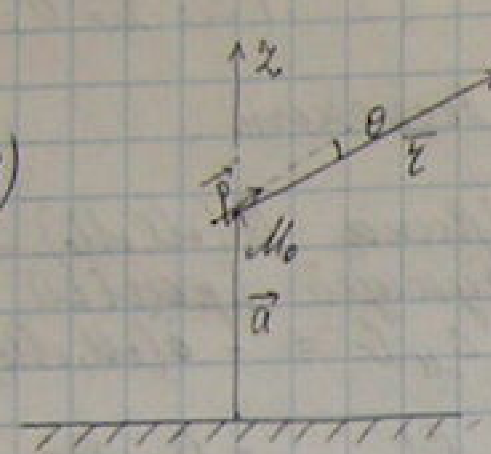
Энергия взаимодействия:

$$U = \frac{1}{2}q\cdot\varphi' = \frac{1}{2}q\left(-\frac{q}{2a}\right) = -\frac{q^2}{4a}.$$

Сила взаимод-я:

$$F = -\frac{\partial U}{\partial a} = -\frac{q^2}{4a^2}.$$

3)



$$\Delta\varphi = -4\pi(\bar p\nabla)\delta(\bar r-\bar a)$$

$$\varphi\Big|_{z=0} = 0.$$

$$\varphi(M) = 4\pi(\bar p\nabla)G(M,M_0) = (\bar p\nabla)\left\{\frac{1}{r} - \frac{1}{r'}\right\} = \frac{(\bar p\bar r)}{r^3} - \frac{(\bar p\cdot\bar r')}{r'^3}.$$

$$\sigma_s = -\frac{1}{4\pi}\frac{\partial\varphi}{\partial z}\Big|_{z=0} = -\frac{1}{4\pi}\left\{+\frac{2p\cos\theta\cdot a}{(\rho^2+a^2)^2} - \frac{2p\cos\theta' a}{(\rho^2+a^2)^2}\right\} =$$

$$= -\frac{1}{2\pi}\frac{pa\cos\theta}{(\rho^2+a^2)^2} + \frac{1}{2\pi}\frac{pa\cos\theta'}{(\rho^2+a^2)^2}.$$

$\bar p = \{p\cos\alpha, 0, p\sin\alpha\}$, $\bar p' = \{-p\cos\alpha, 0, p\sin\alpha\}$

Энергия взаимодействия:

$$U = \frac{(\bar p\bar p')r^2 - 3(\bar p\bar r)(\bar p'\bar r)}{2r^5} = \frac{(-p^2\cos^2\alpha + p^2\sin^2\alpha)r^2 - 3(p\sin\alpha\cdot 2a)(p\sin\alpha\cdot 2a)}{2(2a)^5} = \frac{-4p^2a^2 - 4p^2a^2\cos^2\alpha}{64a^5} =$$

$$= -(1+\cos^2\alpha)\frac{p^2}{16a^3}.$$

Сила: $F_z = -\frac{\partial U}{\partial a} = -(1+\cos^2\alpha)\frac{3p^2}{16a^4}$.

Момент силы: $N_\alpha = -\frac{\partial U}{\partial\alpha} = -\frac{2\cos\alpha\sin\alpha}{16a^3}p^2 = -\frac{p^2\sin 2\alpha}{16a^3}$

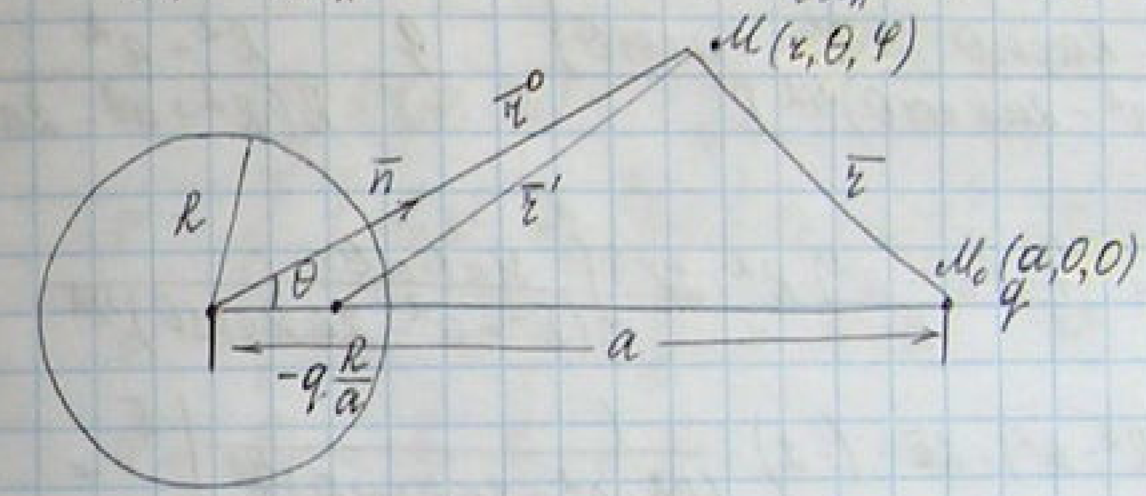
Докажем, что распределение заряда $\rho = -(\bar p'\cdot\nabla)\delta(\bar r)$ описывает элементарный диполь $\bar p'$, помещенный в начале координат.

[$q = -\int(\bar p'\nabla)\delta(\bar r)\,dV = -\oint(\bar p'\bar n)\delta(\bar r)\,dS = 0$, т.к. $\delta(\bar r) = 0$ всюду, кроме $\bar r = 0$.

$$p_\alpha = -\int x_\alpha(\bar p'\nabla)\delta(\bar r)\,dV = -\int x_\alpha p_n'\frac{\partial\delta(\bar r)}{\partial x_n}\,dV =$$

$$= \int p_n'\frac{\partial x_\alpha}{\partial x_n}\delta(\bar r)\,dV = p_n'\frac{\partial x_\alpha}{\partial x_n} = p_n'\delta_{\alpha n} = p_\alpha'.$$]

4)



Введем систему координат (r,θ,φ) в центре шара:

$$\Delta\varphi = -4\pi q\,\delta(\bar r-\bar a),$$

$\varphi\Big|_\Sigma = 0$ (т.к. шар заземлен, Σ - сфера).

Ф-ция Грина: $G(M,M_0) = \frac{1}{4\pi}\left\{\frac{1}{r_{MM_0}} - \frac{R}{a}\frac{1}{r_{MM_1}}\right\}$,

где $M_1 = \left\{\frac{R^2}{a}, 0, 0\right\}$ - точка, сопряженная M_0 отн-но сферы R.

$$\Rightarrow \varphi(M) = 4\pi q\,G(M,M_0) = \frac{q}{r} - \frac{R}{a}\frac{q}{r'}.$$

$$\cancel{\sigma_s = -\frac{1}{4\pi}\frac{\partial\varphi}{\partial n}\Big|_{z=R} = -\frac{1}{4\pi}\left\{-\frac{q(a-z)}{(\rho^2+(a-z)^2)^{3/2}} + \frac{Rq(z-b)}{a(\rho^2+(z-b)^2)^{3/2}}\right\}_{z=R} =}$$

$$\cancel{= \frac{1}{4\pi}\frac{q(a-R)}{(\rho^2+(a-R)^2)^{3/2}} - \frac{1}{4\pi}\frac{qR(R-b)}{a(\rho^2+(R-b)^2)^{3/2}}}$$

$$\sigma_s = -\frac{1}{4\pi}\frac{\partial\varphi}{\partial n}\Big|_\Sigma = -\frac{1}{4\pi}\frac{\partial\varphi}{\partial r_0}\Big|_{r_0=R} = -\frac{1}{4\pi}\left\{\frac{-q(r_0-a\cos\theta)}{(r_0^2+a^2-2r_0a\cos\theta)^{3/2}} + \frac{R}{a}\frac{q(r_0-\frac{R^2}{a}\cos\theta)}{(r_0^2+\frac{R^4}{a^2}-2r_0\frac{R^2}{a}\cos\theta)^{3/2}}\right\}_{r_0=R} = +\frac{1}{4\pi}\left\{\frac{q(R-a\cos\theta)}{(R^2+a^2-2Ra\cos\theta)^{3/2}} -\right.$$

$$\left.-\frac{Rq(R-\frac{R^2}{a}\cos\theta)}{a(R^2+\frac{R^4}{a^2}-2R\frac{R^2}{a}\cos\theta)^{3/2}}\right\} = \frac{1}{4\pi}\left\{\frac{q(R-a\cos\theta)}{(R^2+a^2-2Ra\cos\theta)^{3/2}} -\right.$$

$$\left.-\frac{R^2q(a-R\cos\theta)a^3}{a^2(R^2a^2+R^4-2R^3a\cos\theta)^{3/2}}\right\} = \left\{ \text{—"—} - \frac{aq(a-R\cos\theta)}{R(R^2+a^2-2aR\cos\theta)^{3/2}}\right\}\cdot\frac{1}{4\pi} =$$

$$= \frac{1}{4\pi}\frac{q\{R^2-Ra\cos\theta-a^2+Ra\cos\theta\}}{(R^2+a^2-2aR\cos\theta)^{3/2}\cdot R} = \frac{q}{4\pi}\cdot\frac{R^2-a^2}{R(R^2+a^2-2aR\cos\theta)^{3/2}}$$

$$\sigma = \oint\sigma_s\,ds = \frac{q}{4\pi}\frac{(R^2-a^2)}{R}\cdot R^2\cdot 2\pi\int_0^\pi\frac{\sin\theta\,d\theta}{(R^2+a^2-2aR\cos\theta)^{3/2}} =$$

$$= \frac{q}{4\pi}\frac{R^2}{2aR^2}(R^2-a^2)\cdot 2\pi\cdot(-2)\cdot\frac{1}{(R^2+a^2-2aR\cos\theta)^{1/2}}\Big|_0^\pi =$$

$$= \frac{q}{2a}(R^2-a^2)\left\{\frac{1}{a-R}-\frac{1}{R+a}\right\} = \frac{-q}{2a}\{R+a+R-a\} =$$

$= \dfrac{-qR}{a}$ – индуцированный заряд.

Энергия взаимодействия:

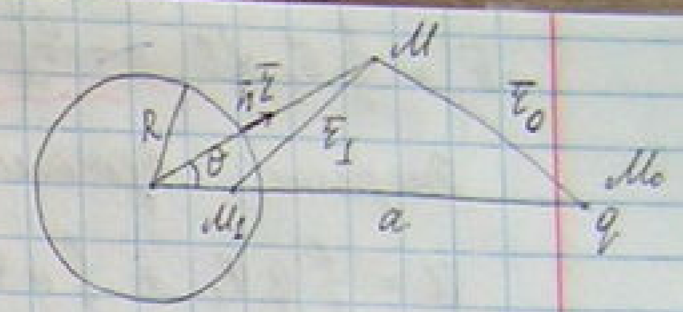
$$U = \frac{1}{2}q\varphi' = \frac{1}{2}q\left(-\frac{qR}{a(a-b)}\right) = -\frac{1}{2}q^2\frac{R}{a(a-\frac{R^2}{a})} =$$

$$= -\frac{1}{2}\cdot\frac{q^2R}{(a^2-R^2)}.$$

Заряд притягивается к шару с силой:

$$F = -\frac{\partial U}{\partial a} = -\frac{q^2Ra}{(a^2-R^2)^2}.$$

$\boxed{5)}$



$$\Delta\varphi = -4\pi q\,\delta(\bar{r}-\bar{a})$$

$$\varphi\big|_\Sigma = \frac{e}{R},\quad \varphi \underset{r\to\infty}{\rightrightarrows} 0$$

$$\varphi = u + v.$$

$$\Delta u = -4\pi q\,\delta(\bar{r}-\bar{a}) \qquad \Delta v = 0.$$

$$u\big|_\Sigma = 0,\ u\underset{r\to\infty}{\rightrightarrows}0. \qquad v\big|_\Sigma = \frac{e}{R},\ v\underset{r\to\infty}{\rightrightarrows}0$$

$$u = 4\pi q\,G(M,M_0) = \frac{q}{r_0} - \frac{R}{a}\frac{q}{r_1};$$

$$v = \sum_{l=1}^{\infty}\left\{A_l r^l + \frac{B_l}{r^{l+1}}\right\}P_l(\cos\theta) \to l=0.$$

$$v = \frac{B_1}{r},\quad v\big|_\Sigma = \frac{e}{R} \Rightarrow B_1 = e.$$

$$\varphi = \frac{q}{r_0} - \frac{R}{a}\cdot\frac{q}{r_1} + \frac{e}{r}.$$

Поверхн. плотность зарядов.

$$\varphi = \frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{R}{a}\cdot\frac{q}{\sqrt{r^2+\frac{R^4}{a^2}-2r\frac{R^2}{a}\cos\theta}} + \frac{e}{r}$$

$$\frac{\partial\varphi}{\partial r}\Big|_{r=R} = \frac{-q(r-a\cos\theta)}{(\sqrt{r^2+a^2-2ra\cos\theta})^3} + \frac{R}{a}\frac{q(r-\frac{R^2}{a}\cos\theta)}{(r^2+\frac{R^4}{a^2}-2r\frac{R^2}{a}\cos\theta)^{3/2}} - \frac{e}{r^2}\Big|_{r=R} =$$

$$= -\frac{q(R-a\cos\theta)}{(R^2+a^2-2Ra\cos\theta)^{3/2}} + \frac{R}{a}\cdot\frac{qRa^3(a-R\cos\theta)}{a(a^2+R^2-2Ra\cos\theta)^{3/2}R^3} - \frac{e}{R^2}$$

$$= -\frac{q(R-a\cos\theta)}{(R^2+a^2-2aR\cos\theta)^{3/2}} + \frac{a}{R}\frac{q(a-R\cos\theta)}{(R^2+a^2-2aR\cos\theta)^{3/2}} - \frac{e}{R^2} =$$

$$= \frac{q\{-R^2+aR\cos\theta+a^2-aR\cos\theta\}}{R(R^2+a^2-2aR\cos\theta)^{3/2}} - \frac{e}{R^2} = \frac{q(a^2-R^2)}{R(R^2+a^2-2aR\cos\theta)^{3/2}} - \frac{e}{R^2}$$

$$\sigma_s = \frac{1}{4\pi R}\frac{q(R^2-a^2)}{(R^2+a^2-2aR\cos\theta)^{3/2}} + \frac{e}{4\pi R^2}.$$

$$\mathcal{E} = \oint \sigma_s \, dS = -\frac{qR}{a} + e.$$

Энергия взаимодействия:

$$U = \frac{1}{2} q\varphi_1 + \frac{1}{2} q_1 \varphi = \frac{q}{2}\left(\frac{-Rq}{a(a-b)}\right) + \frac{q}{2}\left(\frac{e}{a}\right) =$$

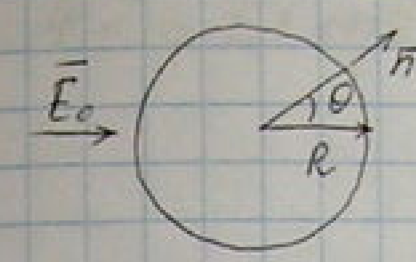
$$= -\frac{q^2}{2}\cdot\frac{R}{a\left(a-\frac{R^2}{a}\right)} + \frac{qe}{2a} = -\frac{q^2R}{(a^2-R^2)} + \frac{qe}{2a}.$$

Сила взаимод-я:

$$F = -\frac{\partial U}{\partial a} = -\frac{2q^2aR}{(a^2-R^2)^2} + \frac{qe}{2a^2}.$$

17. Краевые задачи электростатики

1) Незаряженный проводящий шар



$$\begin{cases} \Delta\varphi = 0, \\ \varphi\big|_{r\to\infty} = -(\bar E_0 \bar r), \\ \varphi\big|_{r=R} = 0. \end{cases}$$

$$\varphi = \sum_{l=0}^{\infty}\left\{A_l r^l + \frac{B_l}{r^{l+1}}\right\}P_l(\cos\theta),$$

$$\varphi\big|_{r\to\infty} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta \Rightarrow l=1,\ A_1 = -E_0,$$

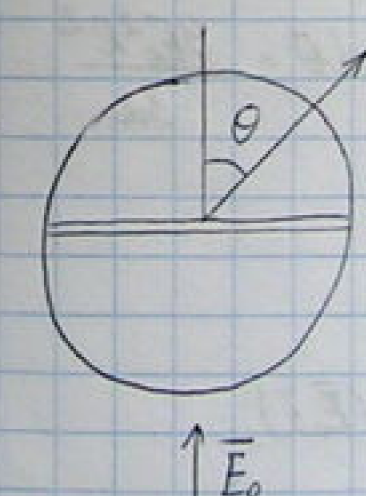
$$\varphi\big|_{r=R} = \left(-E_0R + \frac{B_1}{R^2}\right)\cos\theta = 0 \Rightarrow B_1 = E_0R^3$$

$$\varphi = -(\bar E_0\bar r) + \frac{(\bar E_0\bar r)R^3}{r^3} = -(\bar E_0\bar r) + \frac{(\bar{\mathcal{P}}\bar r)}{r^3}$$

$$\sigma_s = -\frac{1}{4\pi}\frac{\partial\varphi}{\partial r}\Big|_{r=R} = -\frac{1}{4\pi}\left\{-E_0\cos\theta - \frac{2E_0R^3\cos\theta}{R^3}\right\} = \frac{3E_0\cos\theta}{4\pi}$$

$$e = \oint\sigma_s\, dS = \frac{3E_0}{4\pi}\cdot 2\pi R^2\int_0^{\pi}\cos\theta\sin\theta\, d\theta = \frac{3E_0}{4\pi}2\pi R^2\,\frac{\cos^2\theta}{2}\Big|_{\pi}^{0} = 0.$$

$\boxed{2.}$



В щели поле нет.

$$\bar E = -\operatorname{grad}\varphi = -\operatorname{grad}\left\{-E_0 r\cos\theta + \right.$$
$$\left. -\frac{E_0R^3}{r^2}\cos\theta\right\} = +E_0\cos\theta + \frac{2E_0R^3}{r^3}\cos\theta$$

$$E\big|_{\Sigma} = 3E_0\cos\theta$$

Сила, с кой каждое полушарие действует на другое:

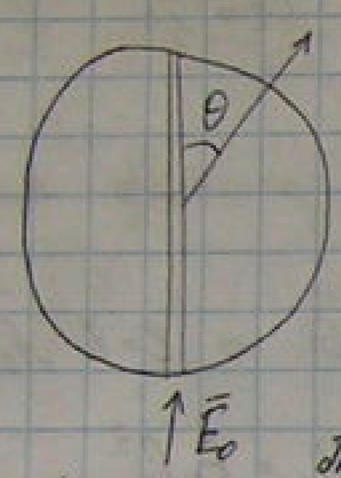
$$F = \frac{1}{8\pi}\oint_S E^2\bar n\, dS.$$

$$F_z = \frac{1}{8\pi}9E_0^2\cdot 2\pi R^2\int_0^{\pi/2}\cos^2\theta\sin\theta\cos\theta\, d\theta = \frac{9E_0^2R^2}{4}\cdot\left\{-\frac{\cos^4\theta}{4}\Big|_0^{\pi/2}\right\} =$$
$$= \frac{9E_0^2R^2}{16}.$$

$$F_1 = \frac{1}{8\pi} 9E_0^2 R^2 \int\limits_0^{\pi/2} \cos^2\theta \sin^2\theta \, d\theta \int\limits_0^{2\pi} \cos\varphi \, d\varphi = 0.$$

$$F_2 = \frac{1}{8\pi} 9E_0^2 R^2 \int\limits_0^{\pi/2} \cos^2\theta \sin^2\theta \, d\theta \int\limits_0^{2\pi} \sin\varphi \, d\varphi = 0.$$

2a)



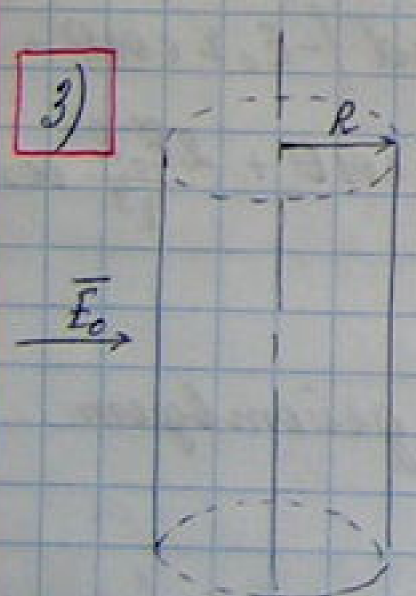
$$F_3 = \frac{1}{8\pi} 9E_0^2 R^2 \int\limits_0^{\pi} d\varphi \int\limits_0^{\pi} \cos^2\theta \cos\theta \sin\theta \, d\theta = 0,$$

$$F_1 = \frac{1}{8\pi} 9E_0^2 R^2 \int\limits_{-\pi/2}^{\pi/2} \cos\varphi \, d\varphi \int\limits_0^{\pi} \cos^2\theta \sin^2\theta \, d\theta =$$

$$= \frac{9E_0^2 R^2}{8\pi} \cdot \sin\varphi \Big|_{-\pi/2}^{\pi/2} \frac{1}{8} \int\limits_0^{\pi} (1 - \cos 4\theta) \, d\theta = \frac{9E_0^2 R^2}{32}.$$

$$F_2 = \frac{1}{8\pi} 9E_0^2 R^2 \int\limits_0^{\pi} \sin\varphi \, d\varphi \int\limits_0^{\pi} \cos^2\theta \sin^2\theta \, d\theta = \frac{9E_0^2 R^2}{32}$$

$\boxed{3)}$



$$\Delta\varphi = 0,$$

$$\varphi\big|_{r\to\infty} = -(\bar{E}_0 \bar{r}),$$

$$\varphi\big|_{r=R} = 0$$

$$\varphi = \sum_{n=0}^{\infty} \left\{ \rho^n (A_n \sin n\varphi + B_n \cos n\varphi) + \right.$$
$$\left. + \rho^{-n} (C_n \sin n\varphi + D_n \cos n\varphi) \right\},$$

$$\varphi\big|_{r\to\infty} = -E_0 r \cos\varphi = B_1 r \cos\varphi, \quad B_1 = -E_0,$$

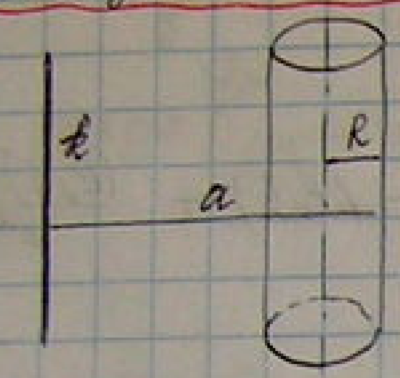
$$\varphi\big|_{r=R} = -E_0 R \cos\theta + R^{-1} (C_1 \sin\varphi + D_1 \cos\varphi) = 0 \Rightarrow$$

$$C_1 = 0, \quad D_1 = E_0 R^2.$$

$$\varphi = -E_0 r \cos\varphi + \frac{E_0 R^2 \cos\varphi}{r} = -(\bar{E}_0 \bar{r}) + \frac{(\bar{E}_0 \bar{r}) R^2}{r^2}.$$

16. Потенциалы и ёмкости.

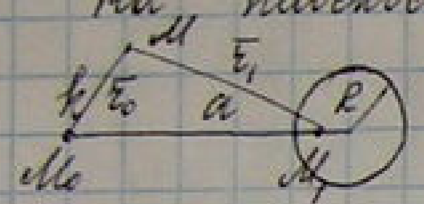
2)



k – лин. плотность.

$$\begin{cases} \Delta\varphi_\ell = -4\pi k\,\delta(\bar\rho - \bar a) \\ \varphi_\ell\big|_{r=R} = 0 \\ \varphi_\ell\big|_{r\to\infty} \Rightarrow 0. \end{cases}$$

На плоскости



$$\varphi_\ell = k4\pi G(M_0, M_1) = 2k\ln\frac{1}{r_0} - 2k\ln\frac{R}{a}\frac{1}{r_1} = 2k\left(\ln\frac{1}{r_0} - \ln\frac{R}{a}\frac{1}{r_1}\right).$$

$$(\sigma_s)_\ell = -\frac{1}{4\pi}\frac{\partial\varphi_\ell}{\partial r}\Big|_{r=R} = -\frac{k2}{4\pi}\Big\{-\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}}\cdot$$

$$\cdot\left(-\frac{1}{2}\right)\frac{2r-2a\cos\theta}{(r^2+a^2-2ra\cos\theta)^{3/2}} + \frac{R}{a}\left(-\frac{1}{2}\right)\frac{2r-2\frac{R^2}{a}\cos\theta}{\left(r^2+\frac{R^4}{a^2}-2\frac{R^2}{a}r\cos\theta\right)^2}\Big\}_{r=R}$$

$$= -\frac{2k}{4\pi}\left\{\frac{R-a\cos\theta}{(R^2+a^2-2Ra\cos\theta)^2} - \frac{R}{a}\frac{a^4\left(R-\frac{R^2}{a}\cos\theta\right)}{R^4(a^2+R^2-2Ra\cos\theta)^2}\right\} =$$

$$= -\frac{k2}{4\pi}\left\{\frac{R-a\cos\theta}{(R^2+a^2-2Ra\cos\theta)^2} - \frac{a^2}{R^2}\cdot\frac{(a-R\cos\theta)}{(a^2+R^2-2Ra\cos\theta)^2}\right\} =$$

$$= -\frac{k2}{4\pi}\cdot\frac{R^3-a^3-(R^2a-a^2R)\cos\theta}{(R^2+a^2-2Ra\cos\theta)^2R^2}$$

Энергия взаимодействия нити и цилиндра:

$$U_\ell = \frac{1}{2}k\cdot\varphi' = \frac{1}{2}k\cdot\left(-2k\ln\frac{R}{a}\frac{1}{\left(a-\frac{R^2}{a}\right)}\right) = -2\frac{k^2}{2}\ln\frac{R}{a^2-R^2}$$

Сила взаимодействия:

$$F_\ell = -\frac{\partial U}{\partial a} = -2\frac{k^2}{2}\cdot\frac{2Ra}{a^2-R^2} = -\frac{2k^2R}{a^2-R^2}.$$

4) Доказать теорему взаимности.

$$\mathrm{div}(\varphi\nabla\varphi') = \varphi\Delta\varphi' + (\nabla\varphi\cdot\nabla\varphi'),$$

$$\mathrm{div}(\varphi'\nabla\varphi) = \varphi'\Delta\varphi + (\nabla\varphi'\nabla\varphi),$$

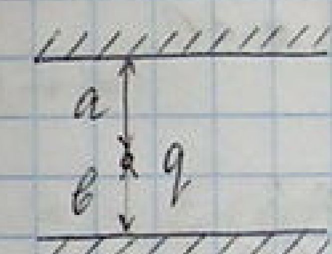
$$\varphi\Delta\varphi' - \varphi'\Delta\varphi = \mathrm{div}\{\varphi\cdot\nabla\varphi' - \varphi'\nabla\varphi\},$$

$$\int_V(\varphi\Delta\varphi' - \varphi'\Delta\varphi)\,dV = \oint_S\left(\varphi\frac{\partial\varphi'}{\partial n} - \varphi'\frac{\partial\varphi}{\partial n}\right)dS,$$

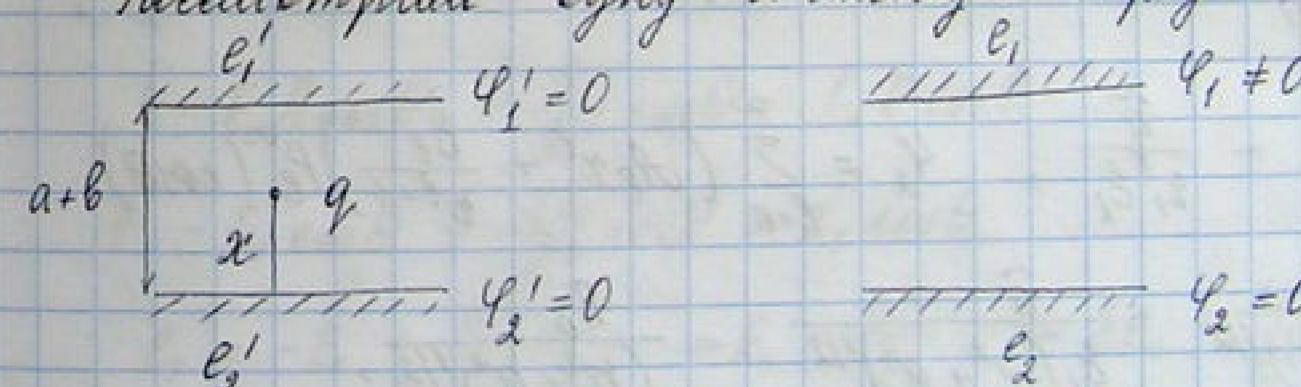
внутри проводников: $\Delta\varphi = 0$, $\Delta\varphi' = 0$,

$$\sum_i\oint_{S_i'}\varphi\frac{\partial\varphi'}{\partial n}\,dS = \sum_i\oint_{S_i}\varphi'\frac{\partial\varphi}{\partial n}\,dS.$$

4a)



Рассмотрим одну систему в разных состояниях:



$$e_1'\varphi_1 + q\cdot\varphi + e_2'\varphi_2 = e_1\varphi_1' + e_2\varphi_2',$$

$$e_1'\varphi_1 + q\cdot\varphi = 0, \quad e_1' = -q\frac{\varphi}{\varphi_1},$$

$$\varphi = \frac{a+b-x}{a+b}\varphi_1, \quad e_1' = -\frac{a}{a+b}q,$$

Т.к. $e_1' + e_2' + q = 0 \Rightarrow e_2' = -q - e_1' = -q + \frac{a}{a+b}q =$

$$= -\frac{b}{a+b}q.$$

Заряды, индуцированные на каждой плоскости: $-\frac{a}{a+b}q, -$

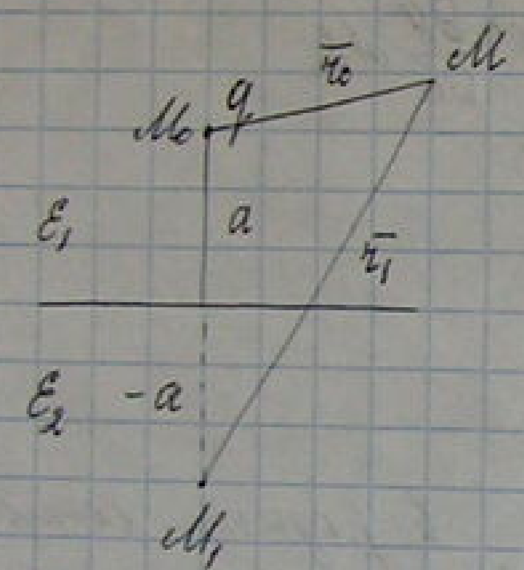
18. Электростатика диэлектриков.

1) Ёмкость длины коаксиального кабеля с внутр радиусом a и внешним – b.

$$C = \frac{q}{\varphi_1 - \varphi_2}; \quad \varphi_l = -2\ln\frac{1}{r} \Rightarrow \varphi_{l\,in} = -2\ln\frac{1}{a}, \quad \varphi_{l\,ex} = -2\ln\frac{1}{b}$$

$$C = \frac{\varkappa l}{2\ln\frac{b}{a}}.$$

3)



$$\Delta\varphi_1 = -\frac{4\pi q}{\varepsilon_1}\delta(\bar r - \bar a), \quad z>0$$

$$\Delta\varphi_2 = 0, \quad z<0$$

$$\varphi_1 = \varphi_2\Big|_{z=0},$$

$$\varepsilon_1\frac{\partial\varphi_1}{\partial n} = \varepsilon_2\frac{\partial\varphi_2}{\partial n}\Big|_{z=0}.$$

$$\varphi_1 = \frac{q}{r_0\varepsilon_1} - \frac{\tilde q}{r_1\varepsilon_1}, \quad \varphi_2 = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l^0(\cos\theta).$$

$$\frac{q}{\varepsilon_1(\rho^2+a^2)^{1/2}} - \frac{\tilde q}{\varepsilon_1(\rho^2+a^2)^{1/2}} = \frac{B_0}{(\rho^2+a^2)^{1/2}},$$

$$+\frac{2qa}{2(\rho^2+a^2)^{3/2}} + \frac{2\tilde q a}{2(\rho^2+a^2)^{3/2}} = \frac{\varepsilon_2 B_0 a}{(\rho^2+a^2)^{3/2}},$$

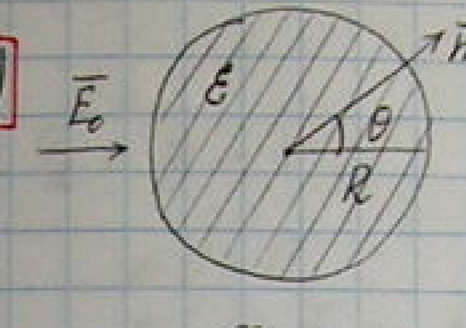
$$B_0 = \frac{q+\tilde q}{\varepsilon_2}, \quad \frac{1}{\varepsilon_1}(q-\tilde q) = \frac{1}{\varepsilon_2}(q+\tilde q),$$

$$q\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_1\varepsilon_2} = \frac{\varepsilon_2+\varepsilon_1}{\varepsilon_1\varepsilon_2}\tilde q \Rightarrow \tilde q = \frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}q,$$

$$\bar B_0 = \frac{1}{\varepsilon_2}q + \frac{1}{\varepsilon_2}\cdot\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}q = \frac{1}{\varepsilon_2}\,\frac{\varepsilon_2+\varepsilon_1+\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}q = \frac{2}{\varepsilon_2+\varepsilon_1}q.$$

$$\varphi_1 = \begin{cases} \dfrac{q}{r_0\varepsilon_1} - \dfrac{1}{\varepsilon_1}\cdot\dfrac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\cdot\dfrac{q}{r_1}, & z>0, \\ \dfrac{2}{\varepsilon_2+\varepsilon_1}\dfrac{q}{r_0}, & z<0. \end{cases}$$

4)



$$\Delta\varphi_1 = 0, \quad r>R,$$

$$\Delta\varphi_2 = 0, \quad r<R.$$

$$\varphi_1 = \varphi_2\Big|_{r=R}, \quad \frac{\partial\varphi_1}{\partial r} = \varepsilon\frac{\partial\varphi_2}{\partial r}\Big|_{r=R}$$

$$\varphi_1 = \sum_{l=1}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l^0(\cos\theta),$$

$$\varphi_2 = \sum_{l=1}^{\infty}\left(C_l r^l + \frac{D_l}{r^{l+1}}\right)P_l^0(\cos\theta).$$

$$\varphi_1\Big|_{r\to\infty} = -E_0 r\cos\theta = A_1 r\cos\theta \Rightarrow A_1 = -E_0.$$

$$\begin{cases} \varphi_1 = -E_0 r\cos\theta + \dfrac{B_1}{r^2}\cos\theta, \\ \varphi_2 = C_1 r\cos\theta. \end{cases}$$

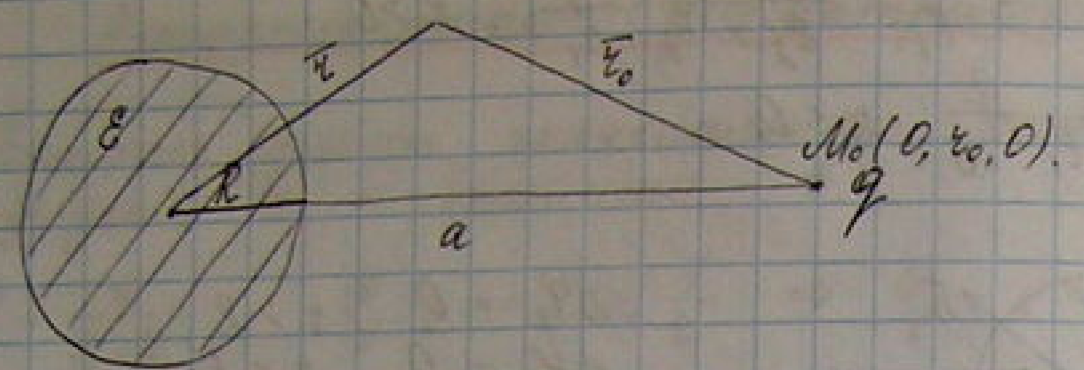
$$\begin{cases} -E_0 R\cos\theta + \dfrac{B_1}{R^2}\cos\theta = C_1 R\cos\theta, \\ -E_0\cos\theta - \dfrac{2B_1}{R^3}\cos\theta = C_1\cos\theta. \end{cases}$$

$$C_1 = -E_0 + \frac{B_1}{R^3}, \quad -E_0 - \frac{2B_1}{R^3} = -\varepsilon E_0 + \frac{\varepsilon B_1}{R^3},$$

$$B_1 = E_0 R^3\frac{\varepsilon-1}{\varepsilon+2}, \quad C_1 = -E_0 + E_0\frac{\varepsilon-1}{\varepsilon+2} = -\frac{3E_0}{\varepsilon+2}$$

$$\varphi = \begin{cases} -(\bar E_0\bar r) + \dfrac{\varepsilon-1}{\varepsilon+2}\dfrac{R^3}{r^3}(\bar E_0\bar r) = -(\bar E_0\bar r) + \dfrac{(\bar p\bar r)}{r^3}, & r>R \\ -\dfrac{3}{\varepsilon+2}(\bar E_0\bar r), & r<R. \end{cases}$$

5)



$$\Delta\varphi_1 = 0, \quad r<R,$$
$$\Delta\varphi_2 = -4\pi q\,\delta(\bar r - \bar a), \quad r>R$$
$$\varphi_1 = \varphi_2\big|_{r=R}, \quad \varepsilon\frac{\partial\varphi_1}{\partial r} = \frac{\partial\varphi_2}{\partial r}$$

$$\varphi_2 = \frac{q}{r_0} + v, \quad \Delta v = 0, \quad r>R.$$

$$\begin{cases} \varphi_2 = \dfrac{q}{r_0} + \sum\limits_{n=1}^{\infty}\left(a_n r^n + \dfrac{b_n}{r^{n+1}}\right)P_n(\cos\theta), \\ \varphi_1 = \sum\limits_{n=0}^{\infty}\left(c_n r^n + \dfrac{d_n}{r^{n+1}}\right)P_n(\cos\theta). \end{cases}$$

Т.к. $\varphi_2\big|_{r\to\infty} \to 0 \Rightarrow a_n = 0 \ \forall n; \quad d_n = 0 \ \forall n.$

$$\frac{q}{r_0} = \frac{q}{a}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n P_n(\cos\theta) = \frac{q}{a} + \frac{q}{a}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n P_n(\cos\theta);$$

$$\begin{cases} \varphi_1 = \sum\limits_{n=0}^{\infty} c_n r^n P_n(\cos\theta), \\ \varphi_2 = \dfrac{q}{a} + \dfrac{q}{a}\sum\limits_{n=1}^{\infty}\left\{\left(\dfrac{r}{a}\right)^n + \dfrac{\tilde b_n}{r^{n+1}}\right\}P_n(\cos\theta), \end{cases} \quad \tilde b_n = \frac{a}{q}b_n.$$

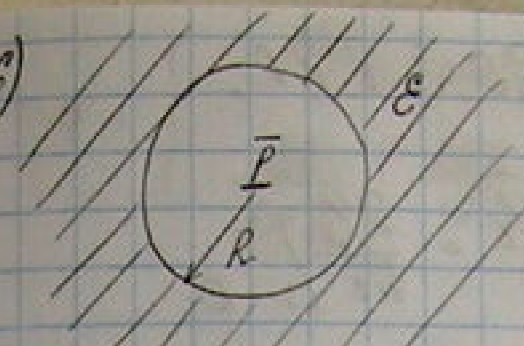
$$\frac{\partial\varphi_1}{\partial r} = \sum_{n=1}^{\infty} c_n n r^{n-1}P_n(\cos\theta); \quad \frac{\partial\varphi_2}{\partial r} = \frac{q}{a}\sum_{n=1}^{\infty}\left\{\frac{n}{a}\left(\frac{r}{a}\right)^{n-1} - \frac{(n+1)\tilde b_n}{r^{n+2}}\right\}P_n(\cos\theta);$$

$$\begin{cases} \dfrac{q}{a}\left\{\left(\dfrac{R}{a}\right)^n + \dfrac{\tilde b_n}{R^{n+1}}\right\} = c_n R^n, \\ \dfrac{q}{a}\left\{\dfrac{n}{a}\left(\dfrac{R}{a}\right)^{n-1} - \dfrac{(n+1)\tilde b_n}{R^{n+2}}\right\} = \varepsilon c_n n R^{n-1}; \end{cases}$$

$$\frac{\varepsilon n}{R}\left\{\left(\frac{R}{a}\right)^n + \frac{\tilde b_n}{R^{n+1}}\right\} = \frac{nR^{n-1}}{a^n} - \frac{(n+1)\tilde b_n}{R^{n+2}},$$

$$\frac{\varepsilon n}{R}\left(\frac{R}{a}\right)^n - \frac{nR^{n-1}}{a^n} = -\tilde b_n\left(\frac{\varepsilon n}{R^{n+2}} + \frac{(n+1)}{R^{n+2}}\right),$$

6)



$$\Delta\varphi_1 = 0, \quad r>R,$$
$$\Delta\varphi_2 = -4\pi(\bar p\nabla)\delta(\bar r), \quad r<R,$$
$$\varphi_1 = \varphi_2\big|_{r=R}, \quad \varepsilon\frac{\partial\varphi_1}{\partial r} = \frac{\partial\varphi_2}{\partial r}\bigg|_{r=R}.$$

$$\varphi_2 = u + v: \quad \Delta u = -4\pi(\bar p\nabla)\delta(\bar r) \Rightarrow u = \frac{(\bar p\bar r)}{r^3}$$

$$\Delta v = 0,$$
$$v\big|_{r=R} = \varphi_2 - \frac{p}{R^2}\cos\theta$$

$$v = \sum_l A_l r^l P_l^{(0)}(\cos\theta), \quad \varphi_1 = \sum_l \frac{B_l}{r^{l+1}}P_l^{(0)}(\cos\theta), \quad l=1.$$

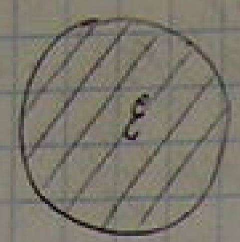
$$\begin{cases} A_1 R\cos\theta + \dfrac{p}{R^2}\cos\theta = \dfrac{B_1}{R^2}\cos\theta, \\ A_1\cos\theta - \dfrac{2p}{R^3}\cos\theta = -\dfrac{2\varepsilon B_1}{R^3}\cos\theta; \end{cases}$$

$$A_1 = \frac{1}{R^3}(B_1 - p), \quad B_1 - 3p = -2\varepsilon B_1$$

$$B_1 = \frac{3p}{1+2\varepsilon}, \quad A_1 = \frac{1}{R^3}\left(\frac{3p}{1+2\varepsilon} - p\right) = \frac{p}{R^3}\,\frac{3-1-2\varepsilon}{1+2\varepsilon} = \frac{2p}{R^3}\cdot\frac{1-\varepsilon}{1+2\varepsilon}$$

$$\varphi = \begin{cases} \dfrac{(\bar p\bar r)}{r^3} + \dfrac{1-\varepsilon}{1+2\varepsilon}\cdot\dfrac{2}{R^3}(\bar p\bar r), & r<R, \\ \dfrac{3}{1+2\varepsilon}\,\dfrac{(\bar p\bar r)}{r^3}, & r>R. \end{cases}$$

7)



$$\Delta \varphi_1 = -4\pi \cancel{\varkappa}\rho\, r\cos\theta, \quad r < R,$$
$$\Delta \varphi_2 = 0, \quad r > R$$
$$\varphi_1 = \varphi_2\Big|_{r=R}, \quad \varepsilon \frac{\partial \varphi_1}{\partial r} = \frac{\partial \varphi_2}{\partial r}\Big|_{r=R}$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\varphi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\frac{\partial\varphi}{\partial\theta}\right) = -4\pi\varkappa r\cos\theta$$

Частное: $\varphi_1^0 = Br^3\cos\theta$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(Br^4\cos\theta\right) + \frac{Br^3}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(-\sin 2\theta\right) = -4\pi\varkappa r\cos\theta,$$

$$12Br\cos\theta - 2Br\cos\theta = -4\pi\cancel{\varkappa}\rho\, r\cos\theta,$$

$$B = -\frac{4\pi\cancel{\varkappa}}{10}\rho = -\frac{2\pi\cancel{\varkappa}}{5}\rho.$$

Общее: $\varphi_1 = \varphi_1^0 + \sum\limits_l A_l r^l P_l(\cos\theta),$

$$\varphi_2 = \sum_l \frac{B_l}{r^{l+1}} P_l(\cos\theta); \qquad l = 1.$$

$$\varphi_1 = -\frac{2\pi\cancel{\varkappa}\rho}{5} r^3\cos\theta + A_1 r\cos\theta,$$

$$\varphi_2 = \frac{B_1}{r^2}\cos\theta;$$

$$\begin{cases} -\dfrac{2\pi\varkappa}{5}R^3\cos\theta + A_1 R\cos\theta = \dfrac{B_1}{R^2}\cos\theta, \\ -3\dfrac{2\pi\cancel{\varkappa}\rho}{5}\varepsilon R^2 + A_1\varepsilon = -\dfrac{2B_1}{R^3}, \end{cases}$$

$$A_1 = -\frac{2B_1}{\varepsilon R^3} + 3\frac{2\pi\cancel{\varkappa}\rho}{5}R^2,$$

$$-\frac{2\pi\cancel{\varkappa}\rho}{5}R^2 - \frac{2B_1}{\varepsilon R^3} + 3\frac{2\pi\cancel{\varkappa}\rho}{5}R^2 = +\frac{B_1}{R^3},$$

$$\frac{B_1}{R^3}\left(1 + 2/\varepsilon\right) = +\frac{2\pi\cancel{\varkappa}\rho}{5}R^2(3-1),$$

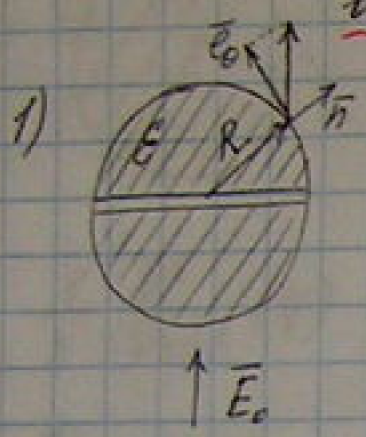
$$B_1 = \frac{2\varepsilon}{2+\varepsilon}R^5\frac{\pi\cancel{\varkappa}\rho}{5}\cdot 2$$

$$A_1 = \frac{3}{5}2\pi\cancel{\varkappa}\rho R^2 - \frac{2\cdot 4\pi\cancel{\varkappa}\rho}{2+\varepsilon}\frac{R^2}{5}$$

$$= \frac{12+6\varepsilon-8}{2+\varepsilon}\cdot\frac{R^2\pi\cancel{\varkappa}\rho}{5} = \frac{2+3\varepsilon}{2+\varepsilon}\cdot\frac{R^2\pi\cancel{\varkappa}\rho}{5}.$$

$$\varphi = \begin{cases} -\dfrac{2\pi\cancel{\varkappa}\rho}{5}r^3\cos\theta + \dfrac{2+3\varepsilon}{2+\varepsilon}\,\dfrac{\pi\cancel{\varkappa}\rho}{5}R^2 r, & r < R, \\ \dfrac{2\varepsilon}{2+\varepsilon}\cdot\dfrac{2\pi\varkappa}{5}\,\dfrac{R^5}{r^2}, & r > R. \end{cases}$$

19. Силы, действующие на диэлектрик во внешнем поле.

1) 

$$\varphi_1 = -(\bar{E}_0 \bar{r})\left\{1 - \frac{A}{r^3}\right\}, \quad r > R$$

$$\varphi_2 = -B(\bar{E}_0 \bar{r}), \quad r < R$$

$$\varphi_1 = \varphi_2\Big|_R, \quad \frac{\partial \varphi_1}{\partial r} = \varepsilon \frac{\partial \varphi_2}{\partial r}\Big|_R$$

$$1 - \frac{A}{R^3} = B, \quad 1 + \frac{2A}{R^3} = \varepsilon B \quad \Rightarrow \quad A = \frac{\varepsilon - 1}{\varepsilon + 2} R^3, \; B = \frac{3}{\varepsilon+2}, \varepsilon \neq 1.$$

$$\bar{E}_2 = \frac{3}{\varepsilon+2}\bar{E}_0, \quad E_{2n} = \frac{3}{\varepsilon+2} E_0 \cos\theta,$$

$$E_{1n} = \varepsilon E_{2n}, \quad E_{1n} = \frac{3\varepsilon}{\varepsilon+2} E_0 \cos\theta,$$

$$E_{2\theta} = -\frac{3}{\varepsilon+2} E_0 \sin\theta. \qquad E_{1t} = E_{2t}$$

$$\bar{E}_1 = E_n \bar{n} + E_\theta \bar{e}_\theta = \frac{3E_0}{\varepsilon+2}\left\{\varepsilon \cos\theta\, \bar{n} - \sin\theta\, \bar{e}_\theta\right\},$$

$$E_n = (\bar{n}\bar{E}), \quad E_\theta = (\bar{E}\bar{e}_\theta)$$

$$\bar{F} = \frac{1}{4\pi}\oint_S \varepsilon\left\{\bar{E}(\bar{n}\bar{E}) - \frac{1}{2}E^2 \bar{n}\right\} dS.$$

$$\bar{E}(\bar{n}\bar{E}) = \left(\frac{3E_0}{\varepsilon+2}\right)^2\left\{\varepsilon^2 \cos^2\theta\, \bar{n} - \varepsilon \sin\theta \cos\theta\, \bar{e}_\theta\right\},$$

$$E^2 = \left(\frac{3E_0}{\varepsilon+2}\right)^2\left\{\varepsilon^2 \cos^2\theta + \sin^2\theta\right\}$$

$$F_i = \frac{1}{4\pi}\oint_S \left\{E_i(\bar{n}\bar{E}) - \frac{1}{2}E^2 n_i\right\} dS$$

$\bar{F}^{(1)} = F^{(1)}\bar{e}_3$ (т/о 1 компонента),

$n_3 = (\bar{n}\bar{e}_3) = \cos\theta, \quad \bar{e}_\theta \bar{e}_3 = -\sin\theta$

$$F_3 = \frac{1}{4\pi}\oint\left\{E_3(\bar{n}\bar{E}) - \frac{1}{2}E^2 n_3\right\} = \frac{1}{4\pi}\cdot 2\pi \cdot \left(\frac{3E_0}{\varepsilon+2}\right)^2 R^2$$

$$\int_0^{\pi/2} \cdot \left\{\frac{\varepsilon^2}{2}\cos^3\theta - \frac{1}{2}\sin^2\theta\cos\theta + \varepsilon \sin^2\theta \cos\theta\right\}\sin\theta\, d\theta =$$

$$= \frac{1}{2}\left(\frac{3E_0}{\varepsilon+2}\right)^2 R^2\left\{\frac{\varepsilon^2}{2}\underbrace{\int_0^{\pi/2}\cos^3\theta \sin\theta\, d\theta}_{1/4} - \frac{1}{2}\underbrace{\int_0^{\pi/2}\sin^3\theta\cos\theta\, d\theta}_{1/4} + \right.$$

$$\left. + \varepsilon \underbrace{\int_0^{\pi/2}\sin^3\theta\cos\theta\, d\theta}_{1/4}\right\}.$$

$$\frac{\varepsilon^2}{8} - \frac{1}{8} + \frac{1}{4}\varepsilon = \frac{1}{8}(\varepsilon^2 + 2\varepsilon - 1)$$

$$F_3^{(1)} = \frac{1}{16}\left(\frac{3E_0}{\varepsilon+2}\right)^2 R^2\left\{\varepsilon^2 + 2\varepsilon - 1\right\}.$$

$$F^{(2)} = \frac{1}{4\pi}\cdot \pi a^2\left\{(\bar{E}\bar{e}_3)(\bar{E}\cdot\bar{n}) - \frac{1}{2}E^2(\bar{n}\bar{e}_3)\right\}$$

$$\bar{e}_3 = -\bar{n}, \quad \bar{n}\bar{e}_3 = -1, \quad (\bar{E}\bar{n}) = -E, \quad (\bar{E}\bar{e}_3) = E,$$

$$(\bar{E}\bar{e}_3)(\bar{E}\bar{n}) = -E^2 \Rightarrow$$

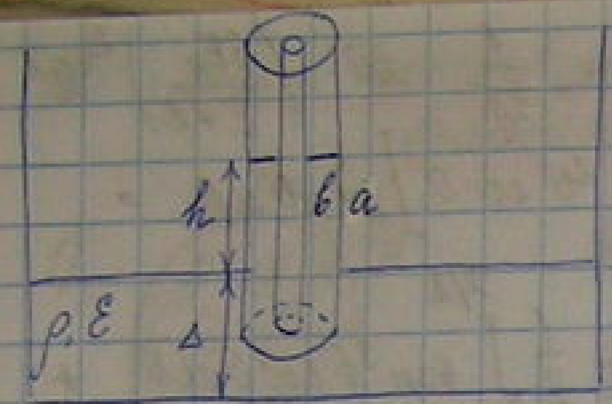
$$F^{(2)} = \frac{1}{4}R^2\left\{-E^2 + \frac{1}{2}E^2\right\} = -\frac{1}{8}R^2E^2 = -\frac{1}{8}\left(\frac{3E_0}{\varepsilon+2}\right)^2 R^2\varepsilon^2.$$

$$F = F^{(1)} + F^{(2)} = \frac{1}{16}\left(\frac{3E_0}{\varepsilon+2}\right)^2 R^2\left\{\varepsilon^2 + 2\varepsilon - 1 - 2\varepsilon^2\right\} =$$

$$= -\frac{1}{16}\left(\frac{3E_0}{\varepsilon+2}\right)^2 R^2\left\{\varepsilon^2 - 2\varepsilon + 1\right\}$$

$$\bar{F} = -\frac{1}{16}\left(\frac{3E_0}{\varepsilon+2}\right)^2 R^2(\varepsilon-1)^2\bar{e}_3.$$

3) а)



Поддержив. пост. φ

Энергия столба Ж:

т.к. $T = const$

$$\tilde{F} = \frac{1}{2}\rho g h^2 \pi (a^2 - b^2) - \frac{C\varphi^2}{2}$$

Для цилиндра: $C = \dfrac{\varepsilon(h+\Delta)}{2\ln\frac{a}{b}} \cdot = \dfrac{\varepsilon(h+\Delta) + l - h - \Delta}{2\ln\frac{a}{b}} =$

$$= \frac{(\varepsilon-1)h}{2\ln\frac{a}{b}} + \frac{(\varepsilon-1)\Delta + l}{2\ln\frac{a}{b}}.$$

Положение равновесия: $\dfrac{\partial \tilde{F}}{\partial h} = 0.$

$$\rho g h \pi (a^2 - b^2) = \frac{\varphi^2}{2} \cdot \frac{\partial C}{\partial h},$$

$$\rho g h \pi (a^2 - b^2) = \frac{\varphi^2}{2} \cdot \frac{(\varepsilon - 1)}{2\ln\frac{a}{b}},$$

$$\boxed{h = \frac{\varepsilon - 1}{4\pi(a^2 - b^2)\rho g \ln\frac{a}{b}} \cdot \varphi^2.}$$

NB. Если бы был задан заряд, то

$$F = \frac{1}{2}\rho g h^2 \pi (a^2 - b^2) + \frac{e^2}{2C},$$

$$\rho g h \pi (a^2 - b^2) = \frac{e^2}{2C^2}\frac{\partial C}{\partial h} \Rightarrow \text{кубич. ур-е.}$$

б) Плоский конденсатор.

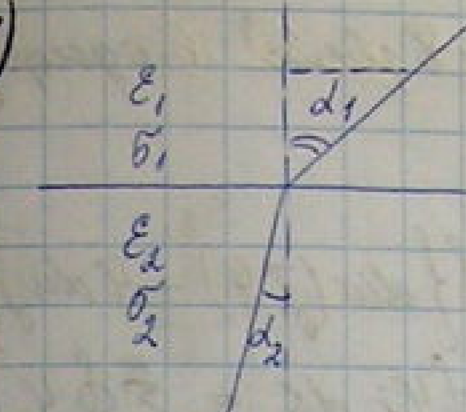
$$a = b + d, \quad (a^2 - b^2)\ln\frac{a}{b} = 2bd \cdot \ln\left|1 + \frac{d}{b}\right| = 2bd \cdot \frac{d}{b} =$$

$$= 2d^2.$$

$$\boxed{h = \frac{(\varepsilon - 1)}{\rho g \cdot 8\pi d^2}\varphi^2 = \frac{(\varepsilon - 1)E^2}{8\pi \rho g}}, \quad \text{т.к. } E = \frac{\varphi}{d}.$$

Задание 20. Стационарные токи в проводника

1)



На границе:

1) $j_{n1} = j_{n2}$, 2) $E_{t1} = E_{t2}$

1) → очевидно; 2) следствие $\mathrm{rot}\,\vec{E} = 0$.

$$E_{t1} = E_{t2} \Leftrightarrow \frac{j_{t1}}{\sigma_1} = \frac{j_{t2}}{\sigma_2}$$

$$\frac{1}{\sigma_1}\frac{j_{t1}}{j_{n1}} = \frac{1}{\sigma_2}\frac{j_{t2}}{j_{n2}} \Rightarrow \boxed{\frac{\mathrm{tg}\,\alpha_1}{\sigma_1} = \frac{\mathrm{tg}\,\alpha_2}{\sigma_2}}$$

закон преломл
линий тока
границе разде

$$\sigma_s' = \sigma_s + \tilde{\sigma}_s.$$

$$\sigma_s = \frac{1}{4\pi}(D_{1n} - D_{2n}) = \frac{1}{4\pi}(\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n}) = \frac{1}{4\pi}\left(\frac{\varepsilon_1}{\sigma_1} - \frac{\varepsilon_2}{\sigma_2}\right) j_n$$

$$-\tilde{\sigma}_s = P_{1n} - P_{2n} = \frac{1}{4\pi}(D_{1n} - E_{1n} - D_{2n} + E_{2n}) = \frac{1}{4\pi}(D_{1n} - D_{2n}) -$$

$$- \frac{1}{4\pi}(E_{1n} - E_{2n}) = \sigma_s - \frac{1}{4\pi}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) j_n$$

$$\sigma_s' = \frac{1}{4\pi}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) j_n.$$

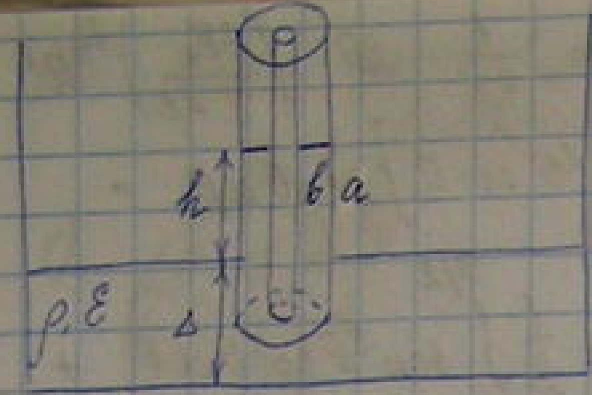
2) $\mathrm{div}\,\vec{D} = 4\pi\rho$

$$\vec{D}(\vec{r}) = \varepsilon(\vec{r}) \cdot \vec{E}(\vec{r}) = \varepsilon(\vec{r}) \cdot \frac{\vec{j}(\vec{r})}{\sigma(\vec{r})}$$

$$\mathrm{div}\,\vec{D}(\vec{r}) = \mathrm{div}\left\{\varepsilon(\vec{r})\frac{\vec{j}(\vec{r})}{\sigma(\vec{r})}\right\} = \left(\nabla, \frac{\varepsilon(\vec{r})\vec{j}(\vec{r})}{\sigma(\vec{r})}\right) = \left(\vec{j}, \nabla\frac{\varepsilon(\vec{r})}{\sigma(\vec{r})}\right)$$

$$+ \frac{\varepsilon(\vec{r})}{\sigma(\vec{r})}(\nabla\vec{j}) = \vec{j}(\vec{r}) \cdot \mathrm{grad}\left(\frac{\varepsilon(\vec{r})}{\sigma(\vec{r})}\right)$$

$$\rho = \frac{1}{4\pi}\vec{j} \cdot \mathrm{grad}\left(\frac{\varepsilon(\vec{r})}{\sigma(\vec{r})}\right).$$

3) а)  Поддерживаем пост. φ

Энергия столба ж.:

т.к. $T = const$

$$\mathcal{F} = \frac{1}{2}\rho g h^2 \pi (a^2 - b^2) - \frac{C\varphi^2}{2}$$

Для цилиндра: $C = \frac{\varepsilon(h+\Delta)}{2\ln\frac{a}{b}} = \frac{\varepsilon(h+\Delta) + l - h - \Delta}{2\ln\frac{a}{b}} =$

$$= \frac{(\varepsilon-1)h}{2\ln\frac{a}{b}} + \frac{(\varepsilon-1)\Delta + l}{2\ln\frac{a}{b}}$$

Положение равновесия: $\frac{\partial \mathcal{F}}{\partial h} = 0$.

$$\rho g h \pi (a^2 - b^2) = \frac{\varphi^2}{2}\cdot\frac{\partial C}{\partial h},$$

$$\rho g h \pi (a^2 - b^2) = \frac{\varphi^2}{2}\cdot\frac{(\varepsilon-1)}{2\ln\frac{a}{b}},$$

$$\boxed{h = \frac{\varepsilon - 1}{4\pi(a^2-b^2)\rho g \ln\frac{a}{b}}\cdot\varphi^2.}$$

NB. Если бы был задан заряд, то

$$\mathcal{F} = \frac{1}{2}\rho g h^2 \pi (a^2 - b^2) + \frac{e^2}{2C},$$

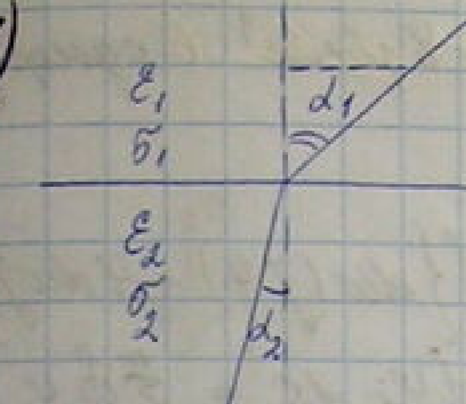
$$\rho g h \pi (a^2 - b^2) = \frac{e^2}{2C^2}\frac{\partial C}{\partial h} \Rightarrow \text{кубич. ур-е.}$$

б) Плоский конденсатор.

$$a = b + d,\quad (a^2 - b^2)\ln\frac{a}{b} = 2bd\cdot\ln\left|1 + \frac{d}{b}\right| = 2bd\cdot\frac{d}{b} = 2d^2.$$

$$\boxed{h = \frac{(\varepsilon-1)}{\rho h\cdot 8\pi d^2}\varphi^2 = \frac{(\varepsilon-1)E^2}{8\pi\rho g}},\quad \text{т.к. } E = \frac{\varphi}{d}.$$

Задание 20. Стационарные токи в проводниках

1) 

На границе:

1) $j_{n1} = j_{n2}$, 2) $E_{t1} = E_{t2}$

1) → очевидно; 2) следствие $\operatorname{rot}\vec{E} = 0$.

$$E_{t1} = E_{t2} \Longleftrightarrow \frac{j_{t1}}{\tilde\sigma_1} = \frac{j_{t2}}{\tilde\sigma_2}$$

$$\frac{1}{\tilde\sigma_1}\frac{j_{t1}}{j_{n1}} = \frac{1}{\tilde\sigma_2}\frac{j_{t2}}{j_{n2}} \Rightarrow \boxed{\frac{\operatorname{tg}\alpha_1}{\tilde\sigma_1} = \frac{\operatorname{tg}\alpha_2}{\tilde\sigma_2}}$$

закон преломления линий тока на границе раздела

$$\sigma_s' = \sigma_s + \tilde\sigma_s.$$

$$\sigma_s = \frac{1}{4\pi}(D_{1n} - D_{2n}) = \frac{1}{4\pi}(\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n}) = \frac{1}{4\pi}\left(\frac{\varepsilon_1}{\tilde\sigma_1} - \frac{\varepsilon_2}{\tilde\sigma_2}\right)j_n$$

$$-\tilde\sigma_s = P_{1n} - P_{2n} = \frac{1}{4\pi}(D_{1n} - E_{1n} - D_{2n} + E_{2n}) = \frac{1}{4\pi}(D_{1n} - D_{2n}) -$$

$$- \frac{1}{4\pi}(E_{1n} - E_{2n}) = \sigma_s - \frac{1}{4\pi}\left(\frac{1}{\tilde\sigma_1} - \frac{1}{\tilde\sigma_2}\right)j_n$$

$$\sigma_s' = \frac{1}{4\pi}\left(\frac{1}{\tilde\sigma_1} - \frac{1}{\tilde\sigma_2}\right)j_n.$$

2) $\operatorname{div}\vec{D} = 4\pi\rho$

$$\vec{D}(\vec r) = \varepsilon(\vec r)\cdot\vec{E}(\vec r) = \varepsilon(\vec r)\cdot\frac{\vec j(\vec r)}{\tilde\sigma(\vec r)}$$

$$\operatorname{div}\vec{D}(\vec r) = \operatorname{div}\left\{\varepsilon(\vec r)\frac{\vec j(\vec r)}{\tilde\sigma(\vec r)}\right\} = \left(\nabla, \frac{\varepsilon(\vec r)\vec j(\vec r)}{\tilde\sigma(\vec r)}\right) = \left(\vec j, \nabla\frac{\varepsilon(\vec r)}{\tilde\sigma(\vec r)}\right) +$$

$$+ \frac{\varepsilon(\vec r)}{\tilde\sigma(\vec r)}(\nabla\vec j) = \vec j(\vec r)\cdot\operatorname{grad}\left(\frac{\varepsilon(\vec r)}{\tilde\sigma(\vec r)}\right)$$

$$\rho = \frac{1}{4\pi}\vec j\cdot\operatorname{grad}\left(\frac{\varepsilon(\vec r)}{\tilde\sigma(\vec r)}\right).$$

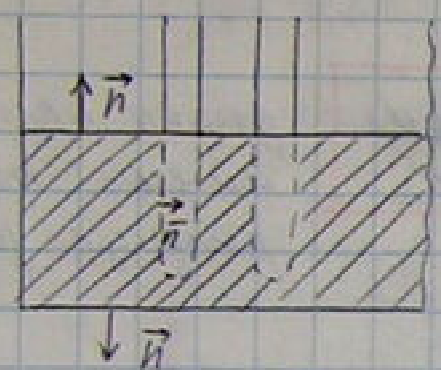
3) Закон Джоуля-Ленца.

Кол-во тепла, выделяющееся в 1с в 1см^3 однород. проводника:

$$(\vec{j}\vec{E}) = \sigma E^2 = \frac{j^2}{\sigma}.$$

$$Q = \int(\vec{j}\vec{E})\,dV = -\int(\vec{j}\,\mathrm{grad}\,\varphi)\,dV = -\int\{\mathrm{div}(\vec{j}\varphi) - \varphi\,\mathrm{div}\,\vec{j}\}\,dV$$

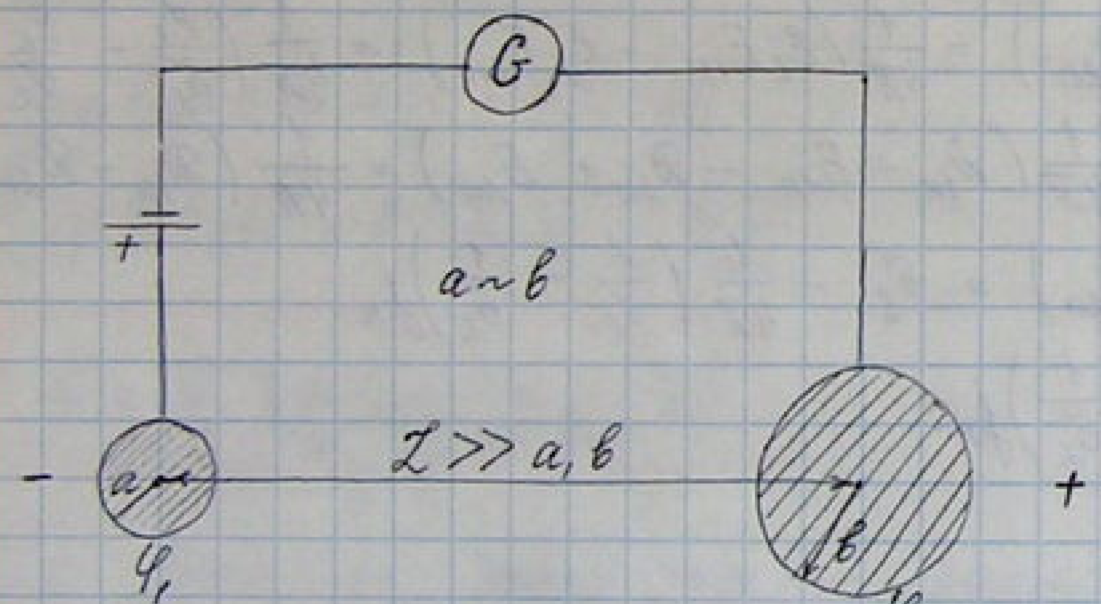
$$= -\int \mathrm{div}(\varphi\vec{j})\,dV = -\oint \varphi(\vec{j}\,d\vec{S}) = -\oint_S \varphi j_n\,dS = \sum_{i=1}^{n}\oint_{S_i}\varphi_i j_i\,dS$$



$$= \sum_{i=1}^{n}\varphi_i\oint_{S_i} j_i\,dS = \sum\varphi_i J_i \quad \left[\frac{\text{эрг}}{c}\right.$$

PS: на внешней границе среды $j_n = 0$, а на пов-тях стержней $\varphi_i = \mathrm{const}$.

4)



Т.к. $L \gg a, b$, то шары не влияют друг на друга.

$$\varphi_1 = \frac{-q}{a}, \quad \varphi_2 = \frac{q}{b}.$$

$$U = \varphi_2 - \varphi_1 = \frac{q}{b} + \frac{q}{a} = \frac{(a+b)q}{ab}$$

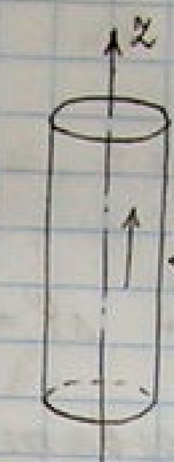
лучай $a = b$.

$a \sim b$: пусть $a = b$, тогда

$$J = 4\pi a^2 j = 4\pi a^2\sigma E = 4\pi a^2\sigma\frac{q}{a^2} = 4\pi\sigma q$$

$$U = RJ = R\cdot 4\pi\sigma q, \quad R = \frac{U}{4\pi\sigma q} = \frac{2q}{4\pi a\sigma q} = \frac{1}{2a\pi\sigma}$$

$\boxed{5)}$



$$\mathrm{rot}\,\vec{H} = \frac{4\pi}{c}\vec{j}, \quad \int(\mathrm{rot}\,\vec{H}\,d\vec{S}) = \frac{4\pi}{c}\int(\vec{j}\,d\vec{S}),$$

NB
μ_0 в пров.,
μ в среде

$$\oint(\vec{H}\,d\vec{l}) = \frac{4\pi}{c}J(r)$$

$$\oint(\vec{H}\,d\vec{l}) = \oint r H_\varphi\,d\varphi = 2\pi r H_\varphi(r).$$

$r \le a$: $\quad 2\pi r H_\varphi(r) = \frac{4\pi}{c} j\pi r^2 = \frac{4\pi}{c}J\left(\frac{r}{a}\right)^2;$

$$H_\varphi = \frac{2Jr}{ca^2};$$

$r > a$: $\quad 2\pi r H_\varphi(r) = \frac{4\pi}{c}J; \quad H_\varphi = \frac{2J}{cr}.$

$\Delta\vec{A} = -\frac{4\pi}{c}\vec{j}\mu_0$. Т.к. $\vec{j} \uparrow\uparrow Oz$, то $\Delta A_x = 0$, $\Delta A_y = 0$,

$$\Delta A_z = -\frac{4\pi}{c}j\mu_0, \text{ где } j = \begin{cases}\frac{J}{\pi a^2}, & r \le a,\\ 0, & r > a.\end{cases}$$

$r \le a$: $\quad \Delta A_z = -\frac{4\pi}{c}j\mu_0 = -\frac{4J}{ca^2}\mu_0 \Leftrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) = -\frac{4\pi}{c}j\mu_0$

$$r\frac{\partial A_z}{\partial r} = -\frac{4J}{ca^2}\mu_0\frac{r^2}{2} = -\frac{2J}{c}\left(\frac{r}{a}\right)^2\mu_0$$

$$A_z = -\frac{2J}{ca^2}\mu\int r\,dr = -\frac{J}{c}\left(\frac{r}{a}\right)^2\mu + C_1,$$

$$A_z(r) = C_1 - \frac{J}{c}\left(\frac{r}{a}\right)^2\mu_0$$

Внутри проводника: $\vec{B} = \mu_0\vec{H} \Rightarrow \mu_0\vec{H} = \mathrm{rot}\,\vec{A}$,

$$\mu\vec{H} = -\frac{\partial A_z}{\partial r}\vec{e}_\varphi, \quad H_\varphi = \frac{2J}{ca^2}r.$$

$r > a$: $\Delta A_z = 0$.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) = 0, \quad r\frac{\partial A_z}{\partial r} = C_2; \quad \frac{\partial A_z}{\partial r}(a) = \frac{C_2}{a} = -\frac{2J}{ca}\mu \Rightarrow C_2 = -\frac{2J}{c}\mu$$

$A_z = -\frac{2J}{c}\mu\ln r + C_3,$

$A_z(a) = -\frac{2J}{c}\mu\ln a + C_3 = C_1 - \frac{J}{c}\mu_0 \Rightarrow C_3 = C_1 - \frac{J}{c}\mu_0 + \frac{2J}{c}\mu\ln a.$

$A_z(r) = C_1 - \frac{J}{c}\left(\mu_0 + 2\mu\ln\frac{r}{a}\right).$

$H_\varphi = \frac{2J}{cr}.$

Вне проводника (где $j=0$) $\vec H = -\text{grad}\,\Psi \Rightarrow \Delta\Psi = 0.$

при условии $\oint_C (\vec H d\vec l) = \frac{4\pi}{c}J$. C - контур, охватив. проводник.

Т.к. $\exists$ т/о H_φ, а $H_r = H_z = 0$, то

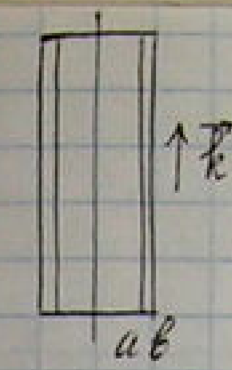
$\vec H = -\text{grad}\,\Psi = -\frac{1}{r}\frac{\partial\Psi}{\partial\varphi}.$

Задача: $\begin{cases} \Delta\Psi = \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\varphi^2} = 0, \\ \oint H_\varphi r\,d\varphi = -\oint_\varphi \frac{\partial\Psi}{\partial\varphi}d\varphi = \frac{4\pi}{c}J. \end{cases}$

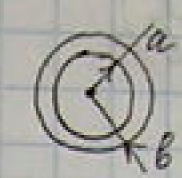
$\frac{\partial\Psi}{\partial\varphi} = C, \quad -C\cdot 2\pi = \frac{4\pi}{c}J \Rightarrow C = -\frac{2J}{c}.$

$\Psi = -\frac{2J}{c}\varphi.$

$H_\varphi = -\frac{1}{r}\frac{\partial\Psi}{\partial\varphi} = \frac{2J}{cr}$

б) 

$H = \frac{2\pi j r}{c}, \quad \vec H = \frac{2\pi}{c}[\vec j\,\vec r]$



$r \le a: \quad \oint(\vec H d\vec l) = \frac{4\pi}{c}\int(\vec j d\vec S) = 0,$

$\vec H = 0.$

$r>a,\ r<b: \quad 2\pi r H_\varphi(r) = \frac{4\pi}{c}j\cdot\pi r^2,$

$H_\varphi = \frac{2\pi}{c}jr = \frac{2\pi}{c}\frac{Jr}{\pi(b^2-a^2)} = \frac{2Jr}{c(b^2-a^2)}$

$r>b: \quad 2\pi r H_\varphi(r) = \frac{4\pi}{c}J, \quad H_\varphi(r) = \frac{2J}{cr}$

$\boxed{NB}$ Н/учесть μ и μ_0

$r<a: \quad \Delta A_z = 0.$

$r\frac{\partial A_z}{\partial r} = C_1, \quad A_z^{(1)} = C_2 + C_1\ln r.$

$a<r<b: \quad \Delta A_z = -\frac{4\pi}{c}j\mu.$

$r\frac{\partial A_z}{\partial r} = -\frac{4\pi}{c}\mu j\frac{r^2}{2}, \quad A_z^{(2)} = C - \frac{\pi\mu j}{c}r^2.$

$r>b: \quad \Delta A_z = 0.$

$r\frac{\partial A_z}{\partial r} = C_3, \quad A_z^{(3)} = C_4 + C_3\ln r.$

$\frac{\partial A_z}{\partial r}(a) = \frac{C_1}{a} = -\frac{2\pi}{c}\mu j a \Rightarrow C_1 = -\frac{2\pi}{c}\mu j a^2.$

$A_z(a) = C_2 - \frac{2\pi}{c}\mu j a^2\ln a = C - \frac{\pi\mu j}{c}a^2,$

$C_2 = C - \frac{\pi\mu j}{c}a^2 + \frac{2\pi}{c}\mu j a^2\ln a$

$\frac{\partial A_z}{\partial r}(b) = \frac{C_3}{b} = -\frac{2\pi}{c}\mu j b \Rightarrow C_3 = -\frac{2\pi}{c}\mu j b^2.$

$A_z(b) = C_4 - \frac{2\pi}{c}\mu j b^2\ln b = C - \frac{\pi\mu j}{c}b^2,$

$C_4 = C - \frac{\pi\mu j}{c}b^2 + \frac{2\pi}{c}\mu j b^2\ln b.$

$r<a: \quad A_z = C - \frac{\pi\mu j a^2}{c}\left(1 + 2\ln\frac{r}{a}\right)$

$a \le r \le b: \quad A_z = C - \frac{\pi\mu j}{c} r^2$

$r > b: \quad A_z = C - \frac{\pi\mu j b^2}{c}\left(1 + 2\ln\frac{r}{b}\right).$

Внутри тонкой проводящей цилиндрич. оболочки рад. b находится коаксиальный провод рад. a, магн. проницу-еть к-ого μ_0. Простр-во м/у проводом и оболочкой заполнено вещ. с μ.

<u>$r \le a$</u> : $\quad \Delta\vec{A} = -\frac{4\pi}{c}\mu_0\vec{j}.$

$\Delta A_z = -\frac{4\pi}{c}\mu_0 j, \qquad \Delta A_x = \Delta A_y = 0.$

$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) = -\frac{4\pi}{c}\mu_0 j,$

$r\frac{\partial A_z}{\partial r} = -\frac{4\pi}{c}\mu_0 j\int_0^r r\,dr = -\frac{4\pi}{c}\mu_0 j\frac{r^2}{2}.$

$\int\partial A_z = -\frac{4\pi}{c}\mu_0 j\int_0^r\frac{r}{2}\,dr = -\frac{4\pi}{c}\mu_0 j\frac{r^2}{4} = -\frac{\pi}{c}\mu_0 j r^2.$

$A_z = C_1 - \frac{\pi}{c}\mu_0 j r^2.$

$H_\varphi = -\frac{\partial A_z}{\partial r} = \frac{2\pi}{c}\mu_0 j r = \frac{2I}{ca}\cdot\frac{r}{a}.$

<u>$a < r \le b$:</u> $\quad \Delta A_z = 0.$

$r\frac{\partial A_z}{\partial r} = C. = -\frac{4\pi}{c}\mu j\frac{a^2}{2}.$

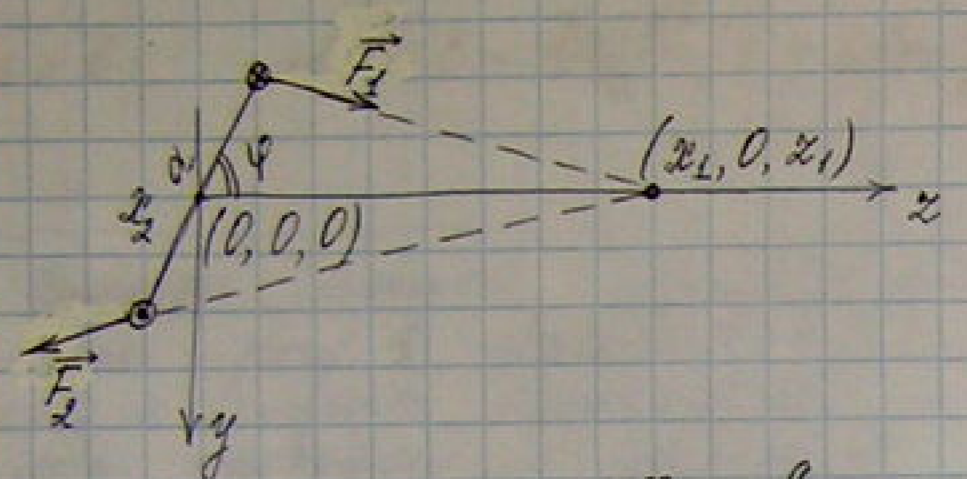
$A_z = -\frac{4\pi}{c}\mu j\frac{a^2}{2}\int_a^r\frac{dr}{r} + \tilde{C} = -\frac{4\pi}{c}\mu j\frac{a^2}{2}\ln\frac{r}{a} + \tilde{C}.$

$r = a: \quad A_z = \tilde{C} = C_1 - \frac{\pi}{c}\mu_0 j a^2.$

$A_z(r) = C_1 - \frac{\pi}{c} j a^2\left(\mu_0 + 2\mu\ln\frac{r}{a}\right).$

$H_\varphi = \frac{2\pi}{c} j a^2\,\frac{1}{r} = \frac{2I}{cr}.$

$\boxed{2)}$



$$L_{12} = \mu \oint\oint \frac{(d\vec{l}_1, d\vec{l}_2)}{r_{12}} = \mu \int_{-\infty}^{+\infty} dx_1 \int_{-\frac{a}{2}}^{\frac{a}{2}} \left\{ \frac{1}{\sqrt{x_1^2 + \left(\frac{a}{2}\sin\varphi\right)^2 + \left(z_1 + \frac{a}{2}\cos\varphi\right)^2}} - \right.$$

$$\left. - \frac{1}{\sqrt{x_1^2 + \left(\frac{a}{2}\sin\varphi\right)^2 + \left(z_1 - \frac{a}{2}\cos\varphi\right)^2}} \right\} dx_2$$

$$I = \int_{-\infty}^{+\infty} dx_1 \int_{-\frac{a}{2}}^{a/2} \frac{dx_2}{\sqrt{x_1^2 + v^2}} = a \cdot \int_{-\infty}^{+\infty} \frac{dx_1}{\sqrt{x_1^2 + v^2}} = 2a \ln\left|x_1 + \sqrt{x_1^2 + v^2}\right| \Big|_0^{\infty}$$

$$L_{12} = 2\mu a \ln \left| \frac{x_1 + \sqrt{x_1^2 + \left(\frac{a}{2}\sin\varphi\right)^2 + \left(z_1 + \frac{a}{2}\cos\varphi\right)^2}}{x_1 + \sqrt{x_1^2 + \left(\frac{a}{2}\sin\varphi\right)^2 + \left(z_1 - \frac{a}{2}\cos\varphi\right)^2}} \right| \Bigg|_0^{\infty} =$$

$$= \mu a \ln \frac{\left(\frac{a}{2}\sin\varphi\right)^2 + \left(z_1 - \frac{a}{2}\cos\varphi\right)^2}{\left(\frac{a}{2}\sin\varphi\right)^2 + \left(z_1 + \frac{a}{2}\cos\varphi\right)^2} = \mu a \ln \frac{4z_1^2 - 4z_1 a\cos\varphi + a^2}{4z_1^2 + 4z_1 a\cos\varphi + a^2}$$

Энергия взаимод-я:

$$U_{12} = \frac{1}{c^2} L_{12} J_1 J_2 = \frac{\mu a J_1 J_2}{c^2} \ln \frac{4z_1^2 - 4z_1 a\cos\varphi + a^2}{4z_1^2 + 4z_1 a\cos\varphi + a^2}.$$

Сила взаимод-я:

$$F_1 = \frac{\partial U}{\partial r_1} = \frac{2\mu a J_1 J_2}{c^2} \cdot \frac{1}{r_1} = \frac{2\mu a J_1 J_2}{c^2} \cdot \frac{1}{\sqrt{\left(\frac{a}{2}\sin\varphi\right)^2 + \left(z_1 - \frac{a}{2}\cos\varphi\right)^2}}$$

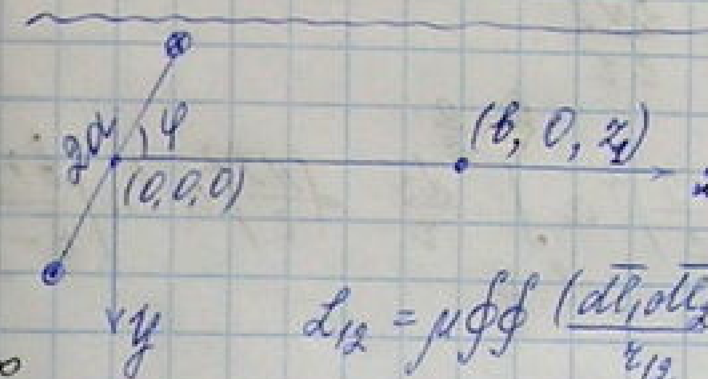
$$r_1 = \sqrt{\left(\frac{a}{2}\sin\varphi\right)^2 + \left(z_1 - \frac{a}{2}\cos\varphi\right)^2}, \quad U(r_1) = \frac{\mu a J_1 J_2}{c^2} \ln \frac{r_1}{\left(\frac{a}{2}\sin\varphi\right)^2 + \left(z_1 + \frac{a}{2}\cos\varphi\right)^2}$$

$$F_2 = \frac{\partial U}{\partial r_2} = -\frac{2\mu a J_1 J_2}{c^2} \cdot \frac{1}{r_2} = \frac{-2\mu a J_1 J_2}{c^2 \sqrt{\left(\frac{a}{2}\sin\varphi\right)^2 + \left(z_1 + \frac{a}{2}\cos\varphi\right)^2}}$$

$$\vec{F} = \vec{F_1} + \vec{F_2}$$

Момент силы:

$$N = \frac{\partial U_{12}}{\partial \varphi} = \mu a \frac{4z_1 a \sin\varphi}{4z_1^2 - 4z_1 a\cos\varphi + a^2} + \mu a \frac{4z_1 a\sin\varphi}{4z_1^2 + 4z_1 a\cos\varphi + a^2}.$$



$$L_{12} = \mu \oint\oint \frac{(d\vec{l}_1, d\vec{l}_2)}{r_{12}} = \mu \int_{-\infty}^{+\infty} dz_1 \int_{-a}^{a} \left\{ \frac{1}{\sqrt{(b - a\cos\varphi)^2 + (a\sin\varphi)^2 + z_1^2}} \right.$$

$$\left. - \frac{1}{\sqrt{(b + a\cos\varphi)^2 + (a\sin\varphi)^2 + z_1^2}} \right\} dz_2 = 2\mu \cdot 2a \ln \left| \frac{z_1 + \sqrt{(b - a\cos\varphi)^2 + (a\sin\varphi)^2 + z_1^2}}{z_1 + \sqrt{(b + a\cos\varphi)^2 + (a\sin\varphi)^2 + z_1^2}} \right| \Bigg|_0^{\infty} = 2a\mu \ln \frac{(a\sin\varphi)^2 + (b + a\cos\varphi)^2}{(a\sin\varphi)^2 + (b - a\cos\varphi)^2} =$$

$$= 2a\mu \ln \frac{b^2 + 2ab\cos\varphi + a^2}{b^2 - 2ab\cos\varphi + a^2}.$$

$$W_{12} = \frac{1}{c^2} L_{12} J_1 J_2 = \frac{1}{c^2} J_1 J_2 \cdot 2a\mu \ln \frac{b^2 + 2ab\cos\varphi + a^2}{b^2 - 2ab\cos\varphi + a^2}$$

Сила: $F_b = \left(\frac{\partial W_{12}}{\partial b}\right)_{J_1, J_2 = const} = \frac{2aJ_1J_2}{c^2}\mu \frac{b^2 - 2ab\cos\varphi + a^2}{b^2 + 2ab\cos\varphi + a^2} \cdot$

$$\cdot \frac{(2b + 2a\cos\varphi)(b^2 - 2ab\cos\varphi + a^2) - (2b - 2a\cos\varphi)(b^2 + 2ab\cos\varphi + a^2)}{(b^2 - 2ab\cos\varphi + a^2)^2} =$$

$$= \frac{2aJ_1J_2\mu}{c^2} \cdot \frac{\cancel{2b^3} + 2ab^2\cos\varphi - 4ab^2\cos\varphi - 4a^2\cancel{b}\cos^2\varphi + \cancel{2a^2b} + 2a^3\cos\varphi}{(a^2 + b^2)^2 - 4a^2b^2\cos^2\varphi}$$

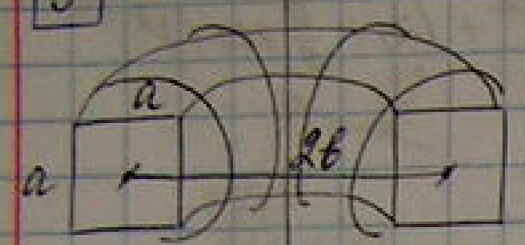
$$\cancel{-2b^3} - 4ab^2\cos\varphi - \cancel{2a^2b} + 2ab^2\cos\varphi + 4a^2\cancel{b}\cos^2\varphi + 2a^3\cos\varphi =$$

$$= \frac{4a\cos\varphi\,(a^2 - b^2)}{(a^2 + b^2)^2 - 4a^2b^2\cos^2\varphi} \cdot \frac{2aJ_1J_2\mu}{c^2}$$

3

$\mu = 1$, N витков

(а)



Если намотка плотная, то снаружи поле нет!

$$\mathcal{F} = \frac{1}{8\pi}\int H^2 dV, \quad \mathcal{F} = \frac{1}{2c^2} L J^2$$

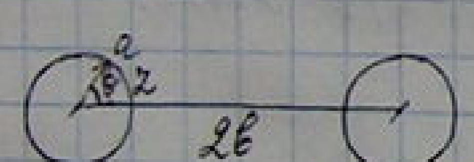
$$\oint (\vec{H} d\vec{l}) = \frac{4\pi}{c} NJ \Rightarrow H_\varphi(r) = \frac{2NJ}{cr}.$$

$$\mathcal{F} = \frac{1}{8\pi}\int_{b-\frac{a}{2}}^{b+\frac{a}{2}} \left(\frac{2NJ}{cr}\right)^2 \cdot 2\pi a r\, dr = \left(\frac{NJ}{c}\right)^2 a \int_{b-\frac{a}{2}}^{b+\frac{a}{2}} \frac{dr}{r} = \left(\frac{NJ}{c}\right)^2 a \ln\left|\frac{b+\frac{a}{2}}{b-\frac{a}{2}}\right| =$$

$$= \left(\frac{NJ}{c}\right)^2 a \ln\left|\frac{2b+a}{2b-a}\right|.$$

$$L = 2N^2 a \ln\left|\frac{2b+a}{2b-a}\right|.$$

(б)



$$z^2 + (r-b)^2 = a^2, \quad z = \pm\sqrt{a^2 - (r-b)^2}.$$

$$dV = 2\pi r \cdot dr\, dz$$

$$\mathcal{F} = \frac{1}{8\pi}\int_{b-a}^{b+a} \left(\frac{2NJ}{cr}\right)^2 \cdot 2\pi r\, dr \int_{-\sqrt{a^2-(r-b)^2}}^{\sqrt{a^2-(r-b)^2}} dz = \left(\frac{NJ}{c}\right)^2 \int_{b-a}^{b+a} \sqrt{a^2-(r-b)^2}\,\frac{dr}{r}$$

$$\mathcal{F} = \frac{1}{8\pi}\int \left(\frac{2NJ}{cr}\right)^2 dV = \frac{N^2J^2}{c^2}\int \frac{z\, dr}{r}$$

Замена: $z = a\sin\theta$, $r = b + a\cos\theta \Rightarrow dr = -a\sin\theta\, d\theta$

$$\mathcal{F} = \frac{N^2J^2}{c^2}\oint \frac{(-a^2)\sin^2\theta\, d\theta}{b + a\cos\theta} \Rightarrow L = 4\pi N^2\left(b - \sqrt{b^2 - a^2}\right).$$

М. поле вне соленоида $= 0$.

4) Найти давление на пов-сть и силу (на единицу длины), растягив. обмотки тороид. соленоида.

$$H_\varphi = \frac{2NJ}{cr}.$$

1 Вычислить коэфт самоиндукции единицы длины коаксиального кабеля. (см. 20.6)

1) $\mathcal{F} = \frac{1}{2c}\mathcal{L} J^2$

2) $\mathcal{F} = \frac{1}{8\pi}\int \mu H^2 dV$.

цск: $dV = r\,dr\,d\varphi\,dz$

$$\mathcal{F}_\ell = \frac{1}{2c^2}\mathcal{L}_\ell J^2$$

$$\mathcal{F}_\ell = \frac{1}{8\pi}\int_0^{2\pi} d\varphi \int \mu H^2 r\,dr = \frac{1}{4}\left\{\mu_0\int_0^a \left(\frac{2J}{ca^2}\right)^2 r^3 dr + \mu\int_a^b \left(\frac{2J}{c}\right)^2 \frac{dr}{r}\right\} = \frac{J^2}{c^2}\left\{\frac{\mu_0}{a^4}\cdot\frac{r^4}{4}\Big|_0^a + \mu\ln r\Big|_a^b\right\} =$$

$$= \frac{J^2}{c^2}\left\{\frac{\mu_0}{4} + \mu\ln\frac{b}{a}\right\} = \frac{J^2}{4c^2}\left(\mu_0 + 4\mu\ln\frac{b}{a}\right).$$

$$\frac{1}{2c^2}\mathcal{L}_\ell J^2 = \frac{J^2}{4c^2}\left(\mu_0 + 4\mu\ln\frac{b}{a}\right).$$

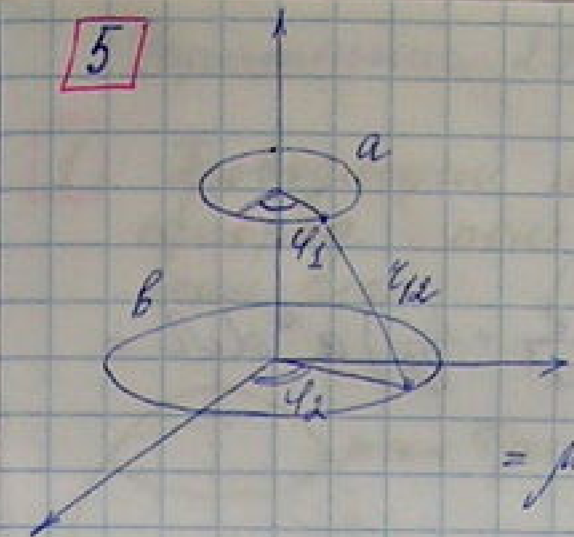
$$\mathcal{L}_\ell = \frac{1}{2}\left(\mu_0 + 4\mu\ln\frac{b}{a}\right).$$

2 $$K = \left(\frac{\partial W_{12}}{\partial\varphi}\right)_{J_1,J_2=const} = \frac{2aJ_1J_2\mu}{c^2}\cdot\frac{b^2-2ab\cos\varphi+a^2}{b^2+2ab\cos\varphi+a^2}\cdot$$

$$\cdot\frac{-2ab\sin\varphi(b^2-2ab\cos\varphi+a^2)-2ab\sin\varphi(b^2+2ab\cos\varphi+a^2)}{(b^2-2ab\cos\varphi+a^2)^2} =$$

$$= \frac{2aJ_1J_2\mu}{c^2}\cdot\frac{-4ab\sin\varphi(a^2+b^2)}{(a^2+b^2)^2-4a^2b^2\cos^2\varphi}$$

5



$h \gg a \sim b \gg r \Rightarrow$

приближение линейных токов.

$$L_{12} = \mu\oint\oint\frac{(d\vec l_1, d\vec l_2)}{r_{12}} = \left|\begin{array}{l} d\vec l_1 = a\,d\varphi_1(-\cos\varphi_1, \sin\varphi_1, 0) \\ d\vec l_2 = b\,d\varphi_2(-\cos\varphi_2, \sin\varphi_2, 0)\end{array}\right.$$

$$= \mu\int_0^{2\pi}d\varphi_1\int_0^{2\pi}d\varphi_2\cdot\frac{(\cos\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2)\,ab}{\sqrt{h^2+(a\cos\varphi_1-b\cos\varphi_2)^2+(a\sin\varphi_1-b\sin\varphi_2)^2}} = 2\pi ab\int_0^{2\pi}\cos\Phi\frac{d\Phi}{\sqrt{h^2+a^2+b^2-2ab\cos\Phi}} \approx$$

$\Phi = \varphi_1 - \varphi_2$

$$\approx \frac{2\pi ab}{h}\int_0^{2\pi}\left(1-\frac{a^2+b^2-2ab\cos\Phi}{2h^2}\right)\cos\Phi\,d\Phi =$$

$$= \frac{2\pi ab}{h}\left\{\underbrace{\int_0^{2\pi}\frac{2h^2-a^2-b^2}{2h^2}\cos\Phi\,d\Phi}_{0} + \frac{ab}{h^2}\int_0^{2\pi}\frac{1-\cos2\Phi}{2}\right\}d\Phi =$$

$$= \frac{2\pi^2a^2b^2}{h^3}.$$

$\boxed{4}$ $l \gg a, b.$

$$\varphi\Big|_{S_a} = \frac{e}{a}, \quad \varphi\Big|_{S_b} = \frac{\tilde{e}}{b}$$

$$J_a = \oint_{S_a} j_n \, dS_a = -\oint_{S_a} \sigma \frac{\partial \varphi}{\partial n} dS_a = \sigma \oint_{S_a} \left(\frac{e}{a^2} + \frac{\tilde{e}}{l^2} \right) a^2 d\Omega =$$

$$= \sigma \left(e + \frac{\tilde{e} a^2}{l^2} \right) 4\pi.$$

$$J_b = \sigma \left(\tilde{e} + \frac{e b^2}{l^2} \right) 4\pi.$$

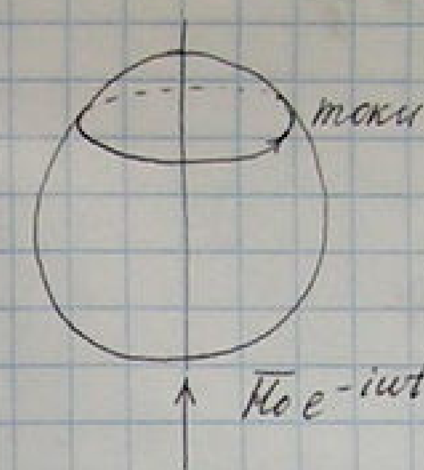
$$J_a + J_b = 0 \Rightarrow \quad e + \frac{\tilde{e} a^2}{l^2} = -\left(\tilde{e} + \frac{e b^2}{l^2} \right),$$

$$e \approx -\tilde{e} = \frac{J}{4\pi\sigma}$$

$$R = \frac{\varphi_a - \varphi_b}{J} = \frac{J}{4\pi\sigma} \cdot \frac{\frac{1}{a} - \frac{1}{l} + \frac{1}{b} - \frac{1}{l}}{J} = \frac{1}{4\pi\sigma} \left(\frac{1}{a} + \frac{1}{b} - \frac{2}{l} \right).$$

Упражнение 23. Скин-эффект.

$\boxed{1}$ Проводящий шар R, σ помещен во внешнее однород. поле $\overline{H}_0 \cos(\omega t)$. Усл. $\delta \ll R \ll \frac{c}{\omega}$, δ - толщина скин-слоя. (т.е. сильный)



Ток Фуко: $\operatorname{rot} \overline{E} = -\frac{1}{c} \frac{\partial \overline{B}}{\partial t}.$

При решении внеш. задачи $H_n = 0$ (скин-слой тоненький, поле не проникает)

$$\overline{M} = \frac{\mu - 1}{\mu + 2} a^3 \overline{H}_0.$$

Т.к. поле не проникает (хотя так не бывает):

$$\overline{M} = -\frac{a^3}{2} \overline{H}_0.$$

Вне $\operatorname{rot} \overline{H} = 0$, поэтому м/ввести скал. пот-л:

$$\overline{H} = -\operatorname{grad} \psi \quad \Rightarrow \quad \begin{cases} \Delta \psi = 0, \\ \left.\frac{\partial \psi}{\partial r}\right|_{r=a} = 0, \\ \left.\overline{H}\right|_{r \to \infty} = -(\overline{H}_0 \overline{r}). \end{cases}$$

$$\psi = -(\overline{H}_0 \overline{r}) + A \frac{(\overline{H}_0 \overline{r})}{r^3}, \quad \text{(NB: внеш. реш. зав. от угла, а внутр. — нет)}$$

$$\psi = -H_0 \left(r - \frac{A}{r^2} \right) \cos\theta,$$

$$\left.\frac{\partial \psi}{\partial r}\right|_{r=a} = -H_0 \left(1 + \frac{2A}{a^3} \right) \cos\theta = 0 \quad \Rightarrow \quad A = -\frac{a^3}{2}$$

$$\psi = -(\overline{H}_0 \overline{r}) - \frac{1}{2} (\overline{H}_0 \overline{r}) \left(\frac{a}{r} \right)^3 = -(\overline{H}_0 \overline{r}) + \frac{(\overline{M} \overline{r})}{r^3} \quad \Rightarrow$$

$$\overline{M} = -\frac{a^3}{2} \overline{H}_0 e^{-i\omega t}$$

$$H_\theta = -\frac{1}{r} \left.\frac{\partial \psi}{\partial \theta}\right|_{r=a} = -H_0 \left(1 - \frac{A}{a^3} \right) \sin\theta = -\frac{3H_0}{2} \sin\theta$$

на пов-сти!

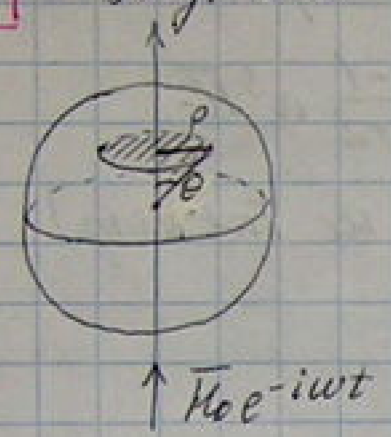
2 Найти тепло, выделяющееся за единицу t в зад. 23.1.

$$Q = \frac{c^2}{32\pi^2\sigma\delta}\oint |H|^2 ds = \frac{9H_0^2a^2}{32\cdot 4\pi^2\sigma\delta}\underbrace{\int \sin^2\theta\, d\Omega}_{4\pi(1-\frac{1}{3}) = \frac{8\pi}{3}} =$$

$$= \frac{3H_0^2a^2}{16\pi\sigma\delta}$$

NB: при сильном скин-эф. $Q \sim \sqrt{\omega}$

3. Зад. 23.1 при условии $R \ll \delta \ll \frac{c}{\omega}$.



Положим $\mu = 1$.

$$\mathrm{rot}\,\overline{E} = -\frac{1}{c}\frac{\partial B}{\partial t},$$

$$\mathrm{rot}\,\overline{E} = \frac{i\omega}{c}\overline{H}.$$

$$2\pi\rho E_\varphi = \frac{i\omega}{c} H \pi\rho^2.$$

$$E_\varphi = \frac{i\omega}{2c}H\rho = \frac{i\omega}{2c}Hr\sin\theta.$$

Тепловые потери:

$$Q = \frac{\sigma}{2}\oint |E|^2 ds = \frac{\sigma}{2}\int \frac{\omega^2}{4c^2}H^2 r^4 dr \sin^2\theta\, d\Omega =$$

$$= \frac{\sigma}{2}\cdot\frac{\omega^2}{4c^2}H^2\frac{r^5}{5}\Big|_0^a\cdot\frac{8\pi}{3} = \frac{\sigma\omega^2\pi}{15c^2}H^2a^5.$$

Здесь $H = \frac{3}{\mu+2}H_0$ - напр. м. поля внутри шара.

(в нулевом {по частоте} приближ.)

$$\Rightarrow Q = \frac{3\pi a^5\sigma\omega^2 H_0}{5c^2(\mu+2)^2}.$$

Положим $\mu = 1$, тогда $H = H_0$!

4